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THREE-DIMENSIONAL STRESS SYSTEMS IN ISOTROPIC PLATES. I

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General analysis is developed for certain three-dimensional stress distributions in a plane plate of infinite extent but of finite thickness which contains a circular cylindrical hole, the faces of the plate being free from applied stress. The analysis is used to solve the problem of a plate under uniform tension in a direction parallel to its faces, the cylindrical hole being free from applied stress. Numerical work is carried out for the case when the diameter of the hole is equal to the thickness of the plate.

1. Introduction

1·1. The problem of finding a complete solution of the elastic equations of equilibrium for stresses in a plane plate is one of considerable analytical difficulty. Many of the difficulties are removed when the problem becomes one of two dimensions which happens for a very wide plate when the state of stress is known as 'plane strain', or for a very thin plate when the state of stress is known as 'plane stress'. Filon (1903) greatly extended the scope of the two-dimensional analysis by his well-known theorem of 'generalized plane stress' in which the value of the stress $\widehat{zz}$ throughout the thickness of a plate whose faces are parallel to the (x, y) plane is neglected and only the *average* values of the remaining stresses are evaluated. This procedure has been widely adopted and great advances have been made in recent years in solving special problems, particularly with the help of complex variable analysis. Some discussion has taken place over the validity of the theory of generalized plane stress. Southwell (1936) pointed out that it was in fact only necessary to assume that the average value $\widetilde{\widehat{zz}}$ of $\widehat{zz}$ taken through the thickness of the plate is zero. Alternative assumptions about $\widehat{zz}$ have been suggested by Green (1945) and Ghosh (1946*a*), but the validity of these assumptions is still unknown. Ghosh (1946*b*), in a slightly more general discussion than that given by Southwell (1936), has shown that a state of stress is possible in a plate with $\widehat{zz}$ identically zero, but the conditions at the edge of the plate cannot then be arbitrarily prescribed; so that in a problem with given boundary conditions it is still not known how near the average stresses found by the generalized plane stress theory are to the actual average values found from a complete three-dimensional solution of the elastic equations.

It is therefore desirable that complete three-dimensional solutions should be obtained at any rate for some of the standard problems dealt with by the generalized plane stress theory, in order to get some accurate estimate of the validity of the theory. Such solutions have also, of course, considerable intrinsic interest from the mathematical point of view. The simplest problem from the point of view of the mathematical analysis appears to be that of an isolated force uniformly distributed through a plate of finite thickness and acting parallel to the faces of the plate. This problem, which has limited physical application, is considered elsewhere. Problems which have considerable physical as well as theoretical interest are those of stress concentrations in plates containing holes of various shapes, and in the present paper analysis is developed for three-dimensional stress systems in a plane plate which is infinitely extended in two dimensions, but which possesses a finite thickness and contains a circular cylindrical hole, the stresses being symmetrical about the plane midway between the faces of the plate and about a plane which is perpendicular to this plane. It is hoped to make extensions to anti-symmetrical systems later.

The analysis is used to solve the particular problem of a plate under uniform tension in a direction parallel to its faces, the cylindrical hole and the faces of the plate being unstressed. This appears to be one of the simplest problems of stress concentration which is not accurately a two-dimensional stress problem.

2. Statement of problem and method of solution

2·1. Consider an isotropic elastic plate of uniform thickness $2h$, bounded by the planes $z = \pm h$ and infinitely extended in the (x, y) plane. The plate contains a circular cylindrical hole of radius a defined by $x^2+y^2 = a^2$. Attention is confined to stress systems in the plate which are symmetrical about the middle plane $z = 0$ and also symmetrical about the plane $y = 0$. The faces of the plate $z = \pm h$ are free from applied normal and shear stress so that

$$\widehat{zz} = \widehat{zx} = \widehat{zy} = 0 \quad (z = \pm h). \tag{2·1·1}$$

The stresses $\widehat{xx}$, $\widehat{yy}$ and $\widehat{xy}$ will usually tend to definite values at infinity. In particular, in the tension problem,

$$\widehat{xx} \to T, \quad \widehat{yy} \to 0, \quad \widehat{xy} \to 0 \quad (x \to \pm\infty), \tag{2·1·2}$$

where T is a constant.

At the surface of the cylindrical hole the stresses $\widehat{rr}$, $\widehat{r\theta}$ and $\widehat{rz}$ have prescribed values. Thus

$$\widehat{rr} = f(\theta, z), \quad \widehat{r\theta} = g(\theta, z), \quad \widehat{rz} = h(\theta, z) \quad (r = a), \tag{2·1·3}$$

where $f(\theta, z)$, $g(\theta, z)$, $h(\theta, z)$ are given functions, all even in the co-ordinate z and symmetrical about $\theta = 0$, (r, θ, z) being cylindrical polar co-ordinates. In the tension problem when the surface of the hole is free from stress f, g, h all vanish.

In a large number of problems the stresses $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{zz}$, $\widehat{rz}$ can each be expressed as the sum of terms of the form

$$\sum_n f_n(r, z) \cos n\theta,$$

while $\widehat{r\theta}$, $\widehat{\theta z}$ can be expressed in the form

$$\sum_n g_n(r, z) \sin n\theta.$$

In the tension problem the stresses reduce to the above forms where the summations for n only extend to $n = 0$ and $n = 2$. It is found that the stresses corresponding to each value of n can be dealt with separately, so attention is directed to stress systems such that $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{zz}$ and $\widehat{rz}$ are each of the form

$$f_n(r, z) \cos n\theta,$$

while $\widehat{r\theta}$, $\widehat{\theta z}$ are of the form

$$g_n(r, z) \sin n\theta.$$

2·2. In order to fulfil the above conditions fundamental solutions of the elastic equations of equilibrium are found which satisfy the boundary conditions (2·1·1) and the conditions at infinity (2·1·2), and these solutions are combined in an infinite series so that the boundary conditions at the edge of the hole may then be satisfied. At the hole $r = a$ the stresses $\widehat{rr}$ and $\widehat{rz}$ can be expanded as Fourier cosine series in z in the range $-h \leqslant z \leqslant h$, and the stress $\widehat{r\theta}$ can be expanded as a Fourier sine series, in general for $-h < z < h$. The boundary conditions (2·1·3) are then satisfied at $r = a$ for all values of z by equating coefficients in the Fourier expansions. Since there are ∞^3+2 terms in the Fourier expansions of $\widehat{rr}$, $\widehat{r\theta}$ and $\widehat{rz}$ at $r = a$,

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it is necessary to find ∞^3+2 independent fundamental solutions of the elastic equations satisfying (2·1·1).

The problem of the convergence of the solution in the form of infinite series is a difficult one and has not been solved. In practice, however, approximations to the complete solution must be made and only a finite number of terms of the series are used.

3. Equations of equilibrium: general solutions

3·1. The equations of equilibrium of a homogeneous isotropic elastic solid when body forces are absent are of the form

$$\left.\begin{aligned}\frac{\partial \widehat{xx}}{\partial x}+\frac{\partial \widehat{xy}}{\partial y}+\frac{\partial \widehat{xz}}{\partial z}&=0,\\ \frac{\partial \widehat{xy}}{\partial x}+\frac{\partial \widehat{yy}}{\partial y}+\frac{\partial \widehat{yz}}{\partial z}&=0,\\ \frac{\partial \widehat{xz}}{\partial x}+\frac{\partial \widehat{yz}}{\partial y}+\frac{\partial \widehat{zz}}{\partial z}&=0,\end{aligned}\right\} \tag{3·1·1}$$

and the components of stress $\widehat{xx}$, $\widehat{yy}$, $\widehat{zz}$, $\widehat{yz}$, $\widehat{zx}$, $\widehat{xy}$ are given in terms of the components of displacement (u_x, u_y, u_z) by the equations

$$\left.\begin{aligned}\widehat{xx}&=\lambda\Delta+2\mu\frac{\partial u_x}{\partial x}, & \widehat{yz}&=\mu\left(\frac{\partial u_z}{\partial y}+\frac{\partial u_y}{\partial z}\right),\\ \widehat{yy}&=\lambda\Delta+2\mu\frac{\partial u_y}{\partial y}, & \widehat{zx}&=\mu\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right),\\ \widehat{zz}&=\lambda\Delta+2\mu\frac{\partial u_z}{\partial z}, & \widehat{xy}&=\mu\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right),\end{aligned}\right\} \tag{3·1·2}$$

where

$$\Delta=\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}, \tag{3·1·3}$$

λ, μ being the elastic constants of Lamé.

If the expressions (3·1·2) are substituted in (3·1·1) the equations of equilibrium take the form

$$\mu\nabla^2(u_x,u_y,u_z)+(\lambda+\mu)\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\Delta=0, \quad \nabla^2\Delta=0. \tag{3·1·4}$$

Formulae for the displacements and stresses will mostly be required in cylindrical polar co-ordinates (r,θ,z), so for reference the necessary results are recorded here. Thus, in terms of the components of displacement (u_r, u_θ, u_z) the stresses are

$$\left.\begin{aligned}\widehat{rr}&=\lambda\Delta+2\mu\frac{\partial u_r}{\partial r}, & \widehat{\theta z}&=\mu\left(\frac{1}{r}\frac{\partial u_z}{\partial\theta}+\frac{\partial u_\theta}{\partial z}\right),\\ \widehat{\theta\theta}&=\lambda\Delta+2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right), & \widehat{rz}&=\mu\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right),\\ \widehat{zz}&=\lambda\Delta+2\mu\frac{\partial u_z}{\partial z}, & \widehat{r\theta}&=\mu\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right).\end{aligned}\right\} \tag{3·1·5}$$

3·2. Some general forms of solutions of equations (3·1·4) have been used by Dougall (1904, 1914) in discussing stresses in plates and cylinders, and these solutions form a convenient starting-point for the present work. For reference these solutions are called here

A, B, C, D. They express the components of displacement and stresses in terms of four harmonic functions ψ, ω, ϕ, χ, i.e. functions which satisfy the equation

$$\nabla^2(\psi, \omega, \phi, \chi) = 0, \tag{3·2·1}$$

and results are given below for both Cartesian and cylindrical polar co-ordinates.

Solution A

$$\left.\begin{aligned} u_x &= 2\frac{\partial\psi}{\partial y}, \quad u_y = -2\frac{\partial\psi}{\partial x}, \quad u_z = 0, \\ u_r &= \frac{2}{r}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = -2\frac{\partial\psi}{\partial r}, \quad \Delta = 0, \end{aligned}\right\} \tag{3·2·2}$$

$$\left.\begin{aligned} &\frac{\widehat{zx}}{2\mu} = \frac{\partial^2\psi}{\partial y\,\partial z}, \quad \frac{\widehat{zy}}{2\mu} = -\frac{\partial^2\psi}{\partial x\,\partial z}, \quad \widehat{zz} = 0, \\ &\frac{\widehat{rz}}{2\mu} = \frac{1}{r}\frac{\partial^2\psi}{\partial\theta\,\partial z}, \quad \frac{\widehat{\theta z}}{2\mu} = -\frac{\partial^2\psi}{\partial r\,\partial z}, \quad \frac{\widehat{r\theta}}{2\mu} = \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}, \\ &\frac{\widehat{rr}}{2\mu} = \frac{2}{r}\frac{\partial^2\psi}{\partial r\,\partial\theta} - \frac{2}{r^2}\frac{\partial\psi}{\partial\theta}, \quad \widehat{rr} + \widehat{\theta\theta} = 0. \end{aligned}\right\} \tag{3·2·3}$$

Solution B

$$\left.\begin{aligned} u_x &= \frac{\partial\omega}{\partial x}, \quad u_y = \frac{\partial\omega}{\partial y}, \quad u_z = \frac{\partial\omega}{\partial z}, \\ u_r &= \frac{\partial\omega}{\partial r}, \quad u_\theta = \frac{1}{r}\frac{\partial\omega}{\partial\theta}, \quad \Delta = 0, \end{aligned}\right\} \tag{3·2·4}$$

$$\left.\begin{aligned} &\frac{\widehat{zx}}{2\mu} = \frac{\partial^2\omega}{\partial x\,\partial z}, \quad \frac{\widehat{zy}}{2\mu} = \frac{\partial^2\omega}{\partial y\,\partial z}, \quad \frac{\widehat{zz}}{2\mu} = \frac{\partial^2\omega}{\partial z^2}, \\ &\frac{\widehat{rz}}{2\mu} = \frac{\partial^2\omega}{\partial r\,\partial z}, \quad \frac{\widehat{\theta z}}{2\mu} = \frac{1}{r}\frac{\partial^2\omega}{\partial\theta\,\partial z}, \quad \frac{\widehat{r\theta}}{2\mu} = \frac{1}{r}\frac{\partial^2\omega}{\partial r\,\partial\theta} - \frac{1}{r^2}\frac{\partial\omega}{\partial\theta}, \\ &\frac{\widehat{rr}}{2\mu} = \frac{\partial^2\omega}{\partial r^2}, \quad \frac{\widehat{rr}+\widehat{\theta\theta}}{2\mu} = -\frac{\partial^2\omega}{\partial z^2}. \end{aligned}\right\} \tag{3·2·5}$$

Solution C

$$\left.\begin{aligned} u_x &= \alpha\frac{\partial\phi}{\partial x} + 2z\frac{\partial^2\phi}{\partial x\,\partial z}, \quad u_y = \alpha\frac{\partial\phi}{\partial y} + 2z\frac{\partial^2\phi}{\partial y\,\partial z}, \\ u_r &= \alpha\frac{\partial\phi}{\partial r} + 2z\frac{\partial^2\phi}{\partial r\,\partial z}, \quad u_\theta = \frac{\alpha}{r}\frac{\partial\phi}{\partial\theta} + \frac{2z}{r}\frac{\partial^2\phi}{\partial\theta\,\partial z}, \\ u_z &= -\alpha\frac{\partial\phi}{\partial z} + 2z\frac{\partial^2\phi}{\partial z^2}, \quad \Delta = 2(1-\alpha)\frac{\partial^2\phi}{\partial z^2}, \end{aligned}\right\} \tag{3·2·6}$$

$$\left.\begin{aligned} &\frac{\widehat{zx}}{2\mu} = \frac{\partial^2\phi}{\partial x\,\partial z} + 2z\frac{\partial^3\phi}{\partial x\,\partial z^2}, \quad \frac{\widehat{zy}}{2\mu} = \frac{\partial^2\phi}{\partial y\,\partial z} + 2z\frac{\partial^3\phi}{\partial y\,\partial z^2}, \\ &\frac{\widehat{rz}}{2\mu} = \frac{\partial^2\phi}{\partial r\,\partial z} + 2z\frac{\partial^3\phi}{\partial r\,\partial z^2}, \quad \frac{\widehat{\theta z}}{2\mu} = \frac{1}{r}\frac{\partial^2\phi}{\partial\theta\,\partial z} + \frac{2z}{r}\frac{\partial^3\phi}{\partial\theta\partial z^2}, \\ &\frac{\widehat{zz}}{2\mu} = -\frac{\partial^2\phi}{\partial z^2} + 2z\frac{\partial^3\phi}{\partial z^3}, \\ &\frac{\widehat{r\theta}}{2\mu} = \frac{\alpha}{r}\frac{\partial^2\phi}{\partial r\,\partial\theta} - \frac{\alpha}{r^2}\frac{\partial\phi}{\partial\theta} + \frac{2z}{r}\frac{\partial^3\phi}{\partial r\,\partial\theta\,\partial z} - \frac{2z}{r^2}\frac{\partial^2\phi}{\partial\theta\partial z}, \\ &\frac{\widehat{rr}}{2\mu} = \alpha\frac{\partial^2\phi}{\partial r^2} + 2z\frac{\partial^3\phi}{\partial r^2\partial z} + (\alpha-3)\frac{\partial^2\phi}{\partial z^2}, \\ &\frac{\widehat{rr}+\widehat{\theta\theta}}{2\mu} = (\alpha-6)\frac{\partial^2\phi}{\partial z^2} - 2z\frac{\partial^3\phi}{\partial z^3}. \end{aligned}\right\} \tag{3·2·7}$$

Solution D

$$\left.\begin{aligned} u_x &= 2x\frac{\partial^2\chi}{\partial z^2}+(\alpha+5)\frac{\partial\chi}{\partial x}, \quad u_y = 2y\frac{\partial^2\chi}{\partial z^2}+(\alpha+5)\frac{\partial\chi}{\partial y}, \\ u_r &= 2r\frac{\partial^2\chi}{\partial z^2}+(\alpha+5)\frac{\partial\chi}{\partial r}, \quad u_\theta = \frac{\alpha+5}{r}\frac{\partial\chi}{\partial\theta}, \quad \Delta = 2(1-\alpha)\frac{\partial^2\chi}{\partial z^2}, \\ u_z &= -2x\frac{\partial^2\chi}{\partial x\partial z}-2y\frac{\partial^2\chi}{\partial y\partial z}+(3-\alpha)\frac{\partial\chi}{\partial z} = -2r\frac{\partial^2\chi}{\partial r\partial z}+(3-\alpha)\frac{\partial\chi}{\partial z}, \end{aligned}\right\} \tag{3·2·8}$$

$$\left.\begin{aligned} \frac{\widehat{zx}}{2\mu} &= x\frac{\partial^3\chi}{\partial z^3}+3\frac{\partial^2\chi}{\partial x\partial z}-x\frac{\partial^3\chi}{\partial x^2\partial z}-y\frac{\partial^3\chi}{\partial x\partial y\partial z}, \\ \frac{\widehat{zy}}{2\mu} &= y\frac{\partial^3\chi}{\partial z^3}+3\frac{\partial^2\chi}{\partial y\partial z}-x\frac{\partial^3\chi}{\partial x\partial y\partial z}-y\frac{\partial^3\chi}{\partial y^2\partial z}, \\ \frac{\widehat{zz}}{2\mu} &= -2x\frac{\partial^3\chi}{\partial x\partial z^2}-2y\frac{\partial^3\chi}{\partial y\partial z^2} = -2r\frac{\partial^3\chi}{\partial r\partial z^2}, \\ \frac{\widehat{rz}}{2\mu} &= 3\frac{\partial^2\chi}{\partial r\partial z}+r\frac{\partial^3\chi}{\partial z^3}-r\frac{\partial^3\chi}{\partial r^2\partial z}, \\ \frac{\widehat{\theta z}}{2\mu} &= \frac{4}{r}\frac{\partial^2\chi}{\partial\theta\partial z}-\frac{\partial^3\chi}{\partial r\partial\theta\partial z}, \\ \frac{\widehat{r\theta}}{2\mu} &= (\alpha+5)\left(\frac{1}{r}\frac{\partial^2\chi}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial\chi}{\partial\theta}\right)+\frac{\partial^3\chi}{\partial\theta\partial z^2}, \\ \frac{\widehat{rr}}{2\mu} &= (\alpha+5)\frac{\partial^2\chi}{\partial r^2}+2r\frac{\partial^3\chi}{\partial r\partial z^2}+(\alpha-1)\frac{\partial^2\chi}{\partial z^2}, \\ \frac{\widehat{rr}+\widehat{\theta\theta}}{2\mu} &= 2r\frac{\partial^3\chi}{\partial r\partial z^2}+(\alpha-7)\frac{\partial^2\chi}{\partial z^2}. \end{aligned}\right\} \tag{3·2·9}$$

The constant α which appears in solutions C and D is related to λ, μ and to Poisson's ratio η by the equations

$$\alpha = \frac{\lambda+3\mu}{\lambda+\mu} = 3-4\eta. \tag{3·2·10}$$

4. Fundamental stress systems

4·1. It is convenient to introduce non-dimensional co-ordinates ρ, ζ defined by*

$$r/h = \rho, \quad z/h = \zeta, \quad a/h = \lambda, \tag{4·1·1}$$

so that the faces of the plate are given by $\zeta = \pm 1$ and the cylindrical hole by $\rho = \lambda$.

The simplest type of solution of the elastic equations containing $\cos n\theta$, $\sin n\theta$ is obtained by putting ω (solution B) in the form

$$\omega_a = a^2(\lambda/\rho)^n \cos n\theta \quad (n \geq 1), \tag{4·1·2}$$

so that, from (3·2·4) and (3·2·5),

$$u_r = -na(\lambda/\rho)^{n+1}\cos n\theta, \quad u_\theta = -na(\lambda/\rho)^{n+1}\sin n\theta, \quad u_z = 0, \tag{4·1·3}$$

$$\frac{\widehat{rr}}{2\mu} = \frac{n(n+1)\,\lambda^{n+2}}{\rho^{n+2}}\cos n\theta, \quad \frac{\widehat{r\theta}}{2\mu} = \frac{n(n+1)\,\lambda^{n+2}}{\rho^{n+2}}\sin n\theta, \quad \widehat{rr}+\widehat{\theta\theta} = 0. \tag{4·1·4}$$

* Since Lamé's constant λ does not appear in the rest of the paper it need not be confused with λ defined in (4·1·1).

This is accurately a plane stress solution and therefore satisfies the boundary conditions (2·1·1) on the faces of the plate. It will be called 'plane stress solution (*a*)'. No new results are obtained by giving similar values to ψ, ϕ, χ.

4·2. The next group of solutions of types *A*, *B* and *C* which are needed are given by

$$\psi = a^2\lambda^{n-2}\left\{\frac{\zeta^2}{\rho^n}+\frac{1}{2(n-1)\,\rho^{n-2}}\right\}\sin n\theta \quad (n\geqslant 2), \tag{4·2·1}$$

$$\left.\begin{aligned} u_r &= 2na\lambda^{n-1}\left\{\frac{\zeta^2}{\rho^{n+1}}+\frac{1}{2(n-1)\,\rho^{n-1}}\right\}\cos n\theta,\\ u_\theta &= 2a\lambda^{n-1}\left\{\frac{n\zeta^2}{\rho^{n+1}}+\frac{n-2}{2(n-1)\,\rho^{n-1}}\right\}\sin n\theta,\\ u_z &= 0,\end{aligned}\right\} \tag{4·2·2}$$

$$\left.\begin{aligned} &\widehat{zz} = 0, && \frac{\widehat{rr}}{2\mu} = -\lambda^n\left\{\frac{2n(n+1)\,\zeta^2}{\rho^{n+2}}+\frac{n}{\rho^n}\right\}\cos n\theta,\\ &\frac{\widehat{rz}}{2\mu} = \frac{2n\lambda^n\zeta}{\rho^{n+1}}\cos n\theta, && \frac{\widehat{r\theta}}{2\mu} = -\lambda^n\left\{\frac{2n(n+1)\,\zeta^2}{\rho^{n+2}}+\frac{n}{\rho^n}\right\}\sin n\theta,\\ &\frac{\widehat{\theta z}}{2\mu} = \frac{2n\lambda^n\zeta}{\rho^{n+1}}\sin n\theta, && \widehat{rr}+\widehat{\theta\theta} = 0.\end{aligned}\right\} \tag{4·2·3}$$

$$\omega = a^2\lambda^{n-2}\left\{\frac{\zeta^2}{\rho^n}+\frac{1}{2(n-1)\,\rho^{n-2}}\right\}\cos n\theta \quad (n\geqslant 2), \tag{4·2·4}$$

$$\left.\begin{aligned} u_r &= -a\lambda^{n-1}\left\{\frac{n\zeta^2}{\rho^{n+1}}+\frac{n-2}{2(n-1)\,\rho^{n-1}}\right\}\cos n\theta,\\ u_\theta &= -na\lambda^{n-1}\left\{\frac{\zeta^2}{\rho^{n+1}}+\frac{1}{2(n-1)\,\rho^{n-1}}\right\}\sin n\theta,\\ u_z &= \frac{2a\lambda^{n-1}\zeta}{\rho^n}\cos n\theta,\end{aligned}\right\} \tag{4·2·5}$$

$$\left.\begin{aligned} &\frac{\widehat{zz}}{2\mu} = \frac{2\lambda^n}{\rho^n}\cos n\theta, && \frac{\widehat{rr}}{2\mu} = \lambda^n\left\{\frac{n(n+1)\,\zeta^2}{\rho^{n+2}}+\frac{n-2}{2\rho^n}\right\}\cos n\theta,\\ &\frac{\widehat{rz}}{2\mu} = -\frac{2n\lambda^n\zeta}{\rho^{n+1}}\cos n\theta, && \frac{\widehat{r\theta}}{2\mu} = n\lambda^n\left\{\frac{(n+1)\,\zeta^2}{\rho^{n+2}}+\frac{1}{2\rho^n}\right\}\sin n\theta,\\ &\frac{\widehat{\theta z}}{2\mu} = -\frac{2n\lambda^n\zeta}{\rho^{n+1}}\sin n\theta, && \frac{\widehat{rr}+\widehat{\theta\theta}}{2\mu} = -\frac{2\lambda^n}{\rho^n}\cos n\theta.\end{aligned}\right\} \tag{4·2·6}$$

$$\phi = a^2\lambda^{n-2}\left\{\frac{\zeta^2}{\rho^n}+\frac{1}{2(n-1)\,\rho^{n-2}}\right\}\cos n\theta \quad (n\geqslant 2), \tag{4·2·7}$$

$$\left.\begin{aligned} u_r &= -a\lambda^{n-1}\left\{\frac{n(\alpha+4)\,\zeta^2}{\rho^{n+1}}+\frac{(n-2)\,\alpha}{2(n-1)\,\rho^{n-1}}\right\}\cos n\theta,\\ u_\theta &= -na\lambda^{n-1}\left\{\frac{(\alpha+4)\,\zeta^2}{\rho^{n+1}}+\frac{\alpha}{2(n-1)\,\rho^{n-1}}\right\}\sin n\theta,\\ u_z &= \frac{(4-2\alpha)\,a\lambda^{n-1}\,\zeta}{\rho^n}\cos n\theta,\end{aligned}\right\} \tag{4·2·8}$$

$$\left.\begin{aligned}&\frac{\widehat{zz}}{2\mu}=-\frac{2\lambda^n}{\rho^n}\cos n\theta, \quad \frac{\widehat{rr}}{2\mu}=\lambda^n\left\{\frac{n(n+1)\,(\alpha+4)\,\zeta^2}{\rho^{n+2}}+\frac{\alpha(n+2)-12}{2\rho^n}\right\}\cos n\theta,\\&\frac{\widehat{rz}}{2\mu}=-\frac{6n\lambda^n\zeta}{\rho^{n+1}}\cos n\theta, \quad \frac{\widehat{r\theta}}{2\mu}=n\lambda^n\left\{\frac{(n+1)\,(\alpha+4)\,\zeta^2}{\rho^{n+2}}+\frac{\alpha}{2\rho^n}\right\}\sin n\theta,\\&\frac{\widehat{\theta z}}{2\mu}=-\frac{6n\lambda^n\zeta}{\rho^{n+1}}\sin n\theta, \quad \frac{\widehat{rr}+\widehat{\theta\theta}}{2\mu}=\frac{2(\alpha-6)\,\lambda^n}{\rho^n}\cos n\theta.\end{aligned}\right\} \tag{4·2·9}$$

The case $n=1$ requires separate treatment. Thus

$$\psi=\frac{a^2}{\lambda}\left\{\frac{\zeta^2}{\rho}-\rho\log\rho\right\}\sin\theta, \tag{4·2·10}$$

$$\left.\begin{aligned}&u_r=2a\left\{\frac{\zeta^2}{\rho^2}-\log\rho\right\}\cos\theta,\\&u_\theta=2a\left\{\frac{\zeta^2}{\rho^2}+\log\rho+1\right\}\sin\theta,\\&u_z=0,\end{aligned}\right\} \tag{4·2·11}$$

$$\left.\begin{aligned}&\widehat{zz}=0, \quad \frac{\widehat{rr}}{2\mu}=-\lambda\left\{\frac{4\zeta^2}{\rho^3}+\frac{2}{\rho}\right\}\cos\theta,\\&\frac{\widehat{rz}}{2\mu}=\frac{2\lambda\zeta}{\rho^2}\cos\theta, \quad \frac{\widehat{r\theta}}{2\mu}=-\frac{4\lambda\zeta^2}{\rho^3}\sin\theta,\\&\frac{\widehat{\theta z}}{2\mu}=\frac{2\lambda\zeta}{\rho^2}\sin\theta, \quad \widehat{rr}+\widehat{\theta\theta}=0.\end{aligned}\right\} \tag{4·2·12}$$

$$\omega=\frac{a^2}{\lambda}\left\{\frac{\zeta^2}{\rho}-\rho\log\rho\right\}\cos\theta, \tag{4·2·13}$$

$$\left.\begin{aligned}&u_r=-a\left\{\frac{\zeta^2}{\rho^2}+\log\rho+1\right\}\cos\theta,\\&u_\theta=-a\left\{\frac{\zeta^2}{\rho^2}-\log\rho\right\}\sin\theta,\\&u_z=\frac{2a\zeta}{\rho}\cos\theta,\end{aligned}\right\} \tag{4·2·14}$$

$$\left.\begin{aligned}&\frac{\widehat{zz}}{2\mu}=\frac{2\lambda}{\rho}\cos\theta, \quad \frac{\widehat{rr}}{2\mu}=\lambda\left\{\frac{2\zeta^2}{\rho^3}-\frac{1}{\rho}\right\}\cos\theta,\\&\frac{\widehat{rz}}{2\mu}=-\frac{2\lambda\zeta}{\rho^2}\cos\theta, \quad \frac{\widehat{r\theta}}{2\mu}=\lambda\left\{\frac{2\zeta^2}{\rho^3}+\frac{1}{\rho}\right\}\sin\theta,\\&\frac{\widehat{\theta z}}{2\mu}=-\frac{2\lambda\zeta}{\rho^2}\sin\theta, \quad \frac{\widehat{rr}+\widehat{\theta\theta}}{2\mu}=-\frac{2\lambda}{\rho}\cos\theta.\end{aligned}\right\} \tag{4·2·15}$$

$$\phi=\frac{a^2}{\lambda}\left\{\frac{\zeta^2}{\rho}-\rho\log\rho\right\}\cos\theta, \tag{4·2·16}$$

$$\left.\begin{aligned}&u_r=-a\left\{\frac{(\alpha+4)\,\zeta^2}{\rho^2}+\alpha\log\rho+\alpha\right\}\cos\theta,\\&u_\theta=-a\left\{\frac{(\alpha+4)\,\zeta^2}{\rho^2}-\alpha\log\rho\right\}\sin\theta,\\&u_z=\frac{(4-2\alpha)\,a\zeta}{\rho}\cos\theta,\end{aligned}\right\} \tag{4·2·17}$$

$$\left.\begin{aligned} \frac{\widehat{zz}}{2\mu} &= -\frac{2\lambda}{\rho}\cos\theta, & \frac{\widehat{rr}}{2\mu} &= \lambda\left\{\frac{2(\alpha+4)\zeta^2}{\rho^3}+\frac{\alpha-6}{\rho}\right\}\cos\theta,\\ \frac{\widehat{rz}}{2\mu} &= -\frac{6\lambda\zeta}{\rho^2}\cos\theta, & \frac{\widehat{r\theta}}{2\mu} &= \lambda\left\{\frac{2(\alpha+4)\zeta^2}{\rho^3}+\frac{\alpha}{\rho}\right\}\sin\theta,\\ \frac{\widehat{\theta z}}{2\mu} &= -\frac{6\lambda\zeta}{\rho^2}\sin\theta, & \frac{\widehat{rr}+\widehat{\theta\theta}}{2\mu} &= \frac{2(\alpha-6)\lambda}{\rho}\cos\theta. \end{aligned}\right\} \tag{4·2·18}$$

A second plane stress solution of the elastic equations which will be called 'plane stress solution (b)' can be found by taking the combination $4\psi+\omega+\phi$ of the stress functions (4·2·1), (4·2·4) and (4·2·7). For reference this will be denoted by

$$4\psi_b+\omega_b+\phi_b, \tag{4·2·19}$$

and the corresponding displacements and stresses for $n\geqslant 2$ are

$$\left.\begin{aligned} u_r &= a\lambda^{n-1}\left\{\frac{(3-\alpha)\,n\zeta^2}{\rho^{n+1}}+\frac{7n+2-\alpha(n-2)}{2(n-1)\,\rho^{n-1}}\right\}\cos n\theta,\\ u_\theta &= a\lambda^{n-1}\left\{\frac{(3-\alpha)\,n\zeta^2}{\rho^{n+1}}+\frac{7n-16-\alpha n}{2(n-1)\,\rho^{n-1}}\right\}\sin n\theta,\\ u_z &= \frac{2(3-\alpha)\,a\lambda^{n-1}\zeta}{\rho^n}\cos n\theta, \end{aligned}\right\} \tag{4·2·20}$$

$$\left.\begin{aligned} \frac{\widehat{rr}}{2\mu} &= \lambda^n\left\{\frac{n(n+1)\,(\alpha-3)\,\zeta^2}{\rho^{n+2}}+\frac{(n+2)\,(\alpha-7)}{2\rho^n}\right\}\cos n\theta,\\ \frac{\widehat{r\theta}}{2\mu} &= \lambda^n\left\{\frac{n(n+1)\,(\alpha-3)\,\zeta^2}{\rho^{n+2}}+\frac{n(\alpha-7)}{2\rho^n}\right\}\sin n\theta,\\ \frac{\widehat{rr}+\widehat{\theta\theta}}{2\mu} &= \frac{2(\alpha-7)\,\lambda^n}{\rho^n}\cos n\theta. \end{aligned}\right\} \tag{4·2·21}$$

The 'plane stress solution (b)' for the case $n=1$, found from (4·2·10), (4·2·13) and (4·2·16), is

$$\left.\begin{aligned} u_r &= a\left\{\frac{(3-\alpha)\,\zeta^2}{\rho^2}-(9+\alpha)\log\rho-1-\alpha\right\}\cos\theta,\\ u_\theta &= a\left\{\frac{(3-\alpha)\,\zeta^2}{\rho^2}+(9+\alpha)\log\rho+8\right\}\sin\theta,\\ u_z &= \frac{2(3-\alpha)\,a\zeta}{\rho}\cos\theta, \end{aligned}\right\} \tag{4·2·22}$$

$$\left.\begin{aligned} \frac{\widehat{rr}}{2\mu} &= \lambda\left\{\frac{2(\alpha-3)\,\zeta^2}{\rho^3}+\frac{\alpha-15}{\rho}\right\}\cos\theta,\\ \frac{\widehat{r\theta}}{2\mu} &= \lambda\left\{\frac{2(\alpha-3)\,\zeta^2}{\rho^3}+\frac{\alpha+1}{\rho}\right\}\sin\theta,\\ \frac{\widehat{rr}+\widehat{\theta\theta}}{2\mu} &= \frac{2(\alpha-7)\,\lambda}{\rho}\cos\theta. \end{aligned}\right\} \tag{4·2·23}$$

Another combination of the three solutions (4·2·1), (4·2·4) and (4·2·7) which is required is

$$(\alpha+1)\,\psi+(\alpha-2)\,\omega+\phi.$$

This will be called a 'plane strain solution', since the displacement u_z vanishes and the remaining displacements and stresses are independent of the co-ordinate z. When $n \geqslant 2$ the displacements and stresses are

$$\left.\begin{aligned} u_r &= \left\{\frac{2\alpha+2(n-1)}{n-1}\right\}\frac{a\lambda^{n-1}}{\rho^{n-1}}\cos n\theta, \\ u_\theta &= \left\{\frac{-2\alpha+2(n-1)}{n-1}\right\}\frac{a\lambda^{n-1}}{\rho^{n-1}}\sin n\theta, \end{aligned}\right\} \tag{4·2·24}$$

$$\left.\begin{aligned} \frac{\widehat{rr}}{2\mu} &= -\frac{2(n+2)\lambda^n}{\rho^n}\cos n\theta, & \frac{\widehat{r\theta}}{2\mu} &= -\frac{2n\lambda^n}{\rho^n}\sin n\theta, \\ \frac{\widehat{rr}+\widehat{\theta\theta}}{2\mu} &= -\frac{8\lambda^n}{\rho^n}\cos n\theta, & \frac{\widehat{zz}}{2\mu} &= -\frac{8\eta\lambda^n}{\rho^n}\cos n\theta, \end{aligned}\right\} \tag{4·2·25}$$

and for $n = 1$,

$$\left.\begin{aligned} u_r &= -2a\{2\alpha\log\rho+\alpha-1\}\cos\theta, \\ u_\theta &= 2a\{2\alpha\log\rho+\alpha+1\}\sin\theta, \end{aligned}\right\} \tag{4·2·26}$$

$$\left.\begin{aligned} \frac{\widehat{rr}}{2\mu} &= -\frac{2(\alpha+3)\lambda}{\rho}\cos\theta, & \frac{\widehat{r\theta}}{2\mu} &= \frac{2(\alpha-1)\lambda}{\rho}\sin\theta, \\ \frac{\widehat{rr}+\widehat{\theta\theta}}{2\mu} &= -\frac{8\lambda}{\rho}\cos\theta, & \frac{\widehat{zz}}{2\mu} &= -\frac{8\eta\lambda}{\rho}\cos\theta. \end{aligned}\right\} \tag{4·2·27}$$

This 'plane strain solution' does not satisfy the boundary conditions (2·1·1) at the faces of the plate. These conditions can, however, be satisfied by adding stresses which tend to zero at infinity and which cancel $\widehat{zz}$ [(4·2·25) and (4·2·27)] on the faces $\zeta = \pm 1$ of the plate without introducing shear stresses $\widehat{zx}$ and $\widehat{zy}$ on these faces. Suitable stresses for this purpose are found from solutions of the types B and C in the form

$$\left.\begin{aligned} \omega &= \cos n\theta\int_0^\infty f(u)\,J_n(u\rho)\cosh u\zeta\,du, \\ \phi &= \cos n\theta\int_0^\infty g(u)\,J_n(u\rho)\cosh u\zeta\,du, \end{aligned}\right\} \tag{4·2·28}$$

where $J_n(u\rho)$ is the Bessel function of the first kind of order n. The arbitrary functions $f(u)$, $g(u)$ have to be chosen so that when $\zeta = \pm 1$ the shear stresses $\widehat{zx}$, $\widehat{zy}$ are zero while the normal stress $\widehat{zz}$ takes the value

$$\frac{\widehat{zz}}{2\mu} = \frac{8\eta\lambda^n}{\rho^n}\cos n\theta \quad (\rho \geqslant \lambda). \tag{4·2·29}$$

Thus, from (3·2·5) and (3·2·7),

$$\left.\begin{aligned} &\int_0^\infty u\{f(u)\sinh u+g(u)(\sinh u+2u\cosh u)\}\,J_n(u\rho)\,du = 0, \\ &\int_0^\infty u^2\{f(u)\cosh u-g(u)(\cosh u-2u\sinh u)\}\,J_n(u\rho)\,du = \frac{8\eta h^2\lambda^n}{\rho^n} \quad (\rho \geqslant \lambda). \end{aligned}\right\} \tag{4·2·30}$$

These integral equations may be solved by using the result (Watson 1944, p. 405)

$$\int_0^\infty \frac{J_n(u\lambda)\,J_n(u\rho)}{u}\,du = \frac{\lambda^n}{2n\rho^n} \quad (\rho \geqslant \lambda, n > 0). \tag{4·2·31}$$

Thus

$$\left.\begin{aligned} f(u)\sinh u+g(u)(\sinh u+2u\cosh u) &= 0, \\ f(u)\cosh u-g(u)(\cosh u-2u\sinh u) &= \frac{16\eta n h^2 J_n(u\lambda)}{u^3}, \end{aligned}\right\} \tag{4·2·32}$$

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and hence

$$\left.\begin{aligned} f(u) &= 16\eta nh^2(\sinh u + 2u\cosh u)\, J_n(u\lambda)/u^3\Sigma, \\ g(u) &= -16\eta nh^2 \sinh u\, J_n(u\lambda)/u^3\Sigma, \end{aligned}\right\} \tag{4·2·33}$$

where

$$\Sigma = 2u + \sinh 2u. \tag{4·2·34}$$

With these values for $f(u)$, $g(u)$ the integrals (4·2·28) are convergent when $n \geqslant 2$ but they diverge at the lower limit for $n = 1$. The divergent terms (which represent a rigid body displacement) are, however, trivial and may be removed by adding suitable functions in the integrands which contribute nothing to the stresses. The final forms for the functions ψ, ω and ϕ, which together give a system of stresses which satisfy the boundary conditions (2·1·1) at the faces of the plate, are

$$\left.\begin{aligned} \psi_0 &= (\alpha+1)\, a^2\lambda^{n-2}\left\{\frac{\zeta^2}{\rho^n} + \frac{1}{2(n-1)\,\rho^{n-2}}\right\}\sin n\theta \quad (n \geqslant 2), \\ &= \frac{(\alpha+1)\, a^2}{\lambda}\left\{\frac{\zeta^2}{\rho} - \rho\log\rho\right\}\sin\theta \quad (n = 1), \end{aligned}\right\} \tag{4·2·35}$$

$$\left.\begin{aligned} \omega_0 &= (\alpha-2)\, a^2\lambda^{n-2}\left\{\frac{\zeta^2}{\rho^n} + \frac{1}{2(n-1)\,\rho^{n-2}}\right\}\cos n\theta \\ &\quad + 16\eta nh^2\cos n\theta \int_0^\infty \frac{(\sinh u + 2u\cosh u)\, J_n(u\lambda)\, J_n(u\rho)\cosh u\zeta}{u^3\Sigma}\, du \quad (n \geqslant 2), \\ &= \frac{(\alpha-2)\, a^2}{\lambda}\left\{\frac{\zeta^2}{\rho} - \rho\log\rho\right\}\cos\theta \\ &\quad + 16\eta h^2\cos\theta \int_0^\infty \left\{\frac{(\sinh u + 2u\cosh u)\, J_1(u\lambda)\, J_1(u\rho)\cosh u\zeta}{u^3\Sigma} - \frac{3\lambda\rho e^{-u}}{16u}\right\} du \quad (n = 1) \end{aligned}\right\} \tag{4·2·36}$$

$$\left.\begin{aligned} \phi_0 &= a^2\lambda^{n-2}\left\{\frac{\zeta^2}{\rho^n} + \frac{1}{2(n-1)\,\rho^{n-2}}\right\}\cos n\theta \\ &\quad - 16\eta nh^2\cos n\theta \int_0^\infty \frac{J_n(u\lambda)\, J_n(u\rho)\sinh u\cosh u\zeta}{u^3\Sigma}\, du \quad (n \geqslant 2), \\ &= \frac{a^2}{\lambda}\left\{\frac{\zeta^2}{\rho} - \rho\log\rho\right\}\cos\theta \\ &\quad - 16\eta h^2\cos\theta \int_0^\infty \left\{\frac{J_1(u\lambda)\, J_1(u\rho)\sinh u\cosh u\zeta}{u^3\Sigma} - \frac{\lambda\rho e^{-u}}{16u}\right\} du \quad (n = 1). \end{aligned}\right\} \tag{4·2·37}$$

These integrals may be differentiated the number of times which are required in order to obtain the displacements and stresses from (3·2·4)–(3·2·7). Thus, for $n \geqslant 2$,

$$\frac{u_r}{\cos n\theta} = \left\{\frac{2\alpha + 2(n-1)}{n-1}\right\}\frac{a\lambda^{n-1}}{\rho^{n-1}} + 16\eta nh\int_0^\infty \frac{J_n(u\lambda)\, J_n'(u\rho)}{u^2\Sigma} \times\{(\sinh u + 2u\cosh u)\cosh u\zeta - \sinh u(\alpha\cosh u\zeta + 2u\zeta\sinh u\zeta)\}\, du, \tag{4·2·38}$$

$$\frac{u_\theta}{\sin n\theta} = \left\{\frac{-2\alpha + 2(n-1)}{n-1}\right\}\frac{a\lambda^{n-1}}{\rho^{n-1}} - \frac{16\eta n^2 h}{\rho}\int_0^\infty \frac{J_n(u\lambda)\, J_n(u\rho)}{u^3\Sigma} \times\{(\sinh u + 2u\cosh u)\cosh u\zeta - \sinh u(\alpha\cosh u\zeta + 2u\zeta\sinh u\zeta)\}\, du, \tag{4·2·39}$$

$$\frac{u_z}{\cos n\theta} = 16\eta nh\int_0^\infty \frac{J_n(u\lambda)\, J_n(u\rho)}{u^2\Sigma} \times\{(\sinh u + 2u\cosh u)\sinh u\zeta + \sinh u(\alpha\sinh u\zeta - 2u\zeta\cosh u\zeta)\}\, du, \tag{4·2·40}$$

and when $n = 1$,

$$\frac{u_r}{\cos\theta} = -2a(2\alpha\log\rho+\alpha-1) + 16\eta h\int_0^\infty \left[\frac{J_1(u\lambda)\,J_1'(u\rho)}{u^2\Sigma}\{(\sinh u+2u\cosh u)\cosh u\zeta - \sinh u(\alpha\cosh u\zeta+2u\zeta\sinh u\zeta)\} - \frac{\eta\lambda e^{-u}}{4u}\right]du, \tag{4·2·41}$$

$$\frac{u_\theta}{\sin\theta} = 2a(2\alpha\log\rho+\alpha+1) - \frac{16\eta h}{\rho}\int_0^\infty \left[\frac{J_1(u\lambda)\,J_1(u\rho)}{u^3\Sigma}\{(\sinh u+2u\cosh u)\cosh u\zeta - \sinh u(\alpha\cosh u\zeta+2u\zeta\sinh u\zeta)\} - \frac{\eta\lambda\rho e^{-u}}{4u}\right]du, \tag{4·2·42}$$

the formula (4·2·40) for u_z being valid for $n = 1$. Also

$$\frac{\widehat{rr}}{2\mu\cos n\theta} = 16\eta n\int_0^\infty \frac{J_n(u\lambda)}{u\Sigma}[\{\{(1-\alpha)\sinh u+2u\cosh u\}\cosh u\zeta - 2u\zeta\sinh u\sinh u\zeta\}J_n''(u\rho)+(3-\alpha)\sinh u\cosh u\zeta J_n(u\rho)]\,du - \begin{Bmatrix} \dfrac{2(n+2)\lambda^n}{\rho^n} & (n\geqslant 2) \\ \dfrac{2(\alpha+3)\lambda}{\rho} & (n=1) \end{Bmatrix}, \tag{4·2·43}$$

$$\frac{\widehat{r\theta}}{2\mu\sin n\theta} = \frac{16\eta n^2}{\rho^2}\int_0^\infty \frac{J_n(u\lambda)\{J_n(u\rho)-u\rho J_n'(u\rho)\}}{u^3\Sigma} \times[\{(1-\alpha)\sinh u+2u\cosh u\}\cosh u\zeta-2u\zeta\sinh u\sinh u\zeta]\,du - \begin{Bmatrix} \dfrac{2n\lambda^n}{\rho^n} & (n\geqslant 2) \\ \dfrac{2(1-\alpha)\lambda}{\rho} & (n=1) \end{Bmatrix}, \tag{4·2·44}$$

and, for $n\geqslant 1$,

$$\frac{\widehat{rz}}{2\mu\cos n\theta} = 32\eta n\int_0^\infty \frac{J_n(u\lambda)\,J_n'(u\rho)}{\Sigma}\{\cosh u\sinh u\zeta-\zeta\sinh u\cosh u\zeta\}\,du, \tag{4·2·45}$$

$$\frac{\widehat{\theta z}}{2\mu\sin n\theta} = -\frac{32\eta n^2}{\rho}\int_0^\infty \frac{J_n(u\lambda)\,J_n(u\rho)}{u\Sigma}\{\cosh u\sinh u\zeta-\zeta\sinh u\cosh u\zeta\}\,du, \tag{4·2·46}$$

$$\frac{\widehat{zz}}{2\mu\cos n\theta} = -\frac{8\eta\lambda^n}{\rho^n}+32\eta n\int_0^\infty \frac{J_n(u\lambda)\,J_n(u\rho)}{u\Sigma} \times\{(\sinh u+u\cosh u)\cosh u\zeta-u\zeta\sinh u\sinh u\zeta\}\,du, \tag{4·2·47}$$

$$\frac{\widehat{rr}+\widehat{\theta\theta}}{2\mu\cos n\theta} = -\frac{8\lambda^n}{\rho^n}-16\eta n\int_0^\infty \frac{J_n(u\lambda)\,J_n(u\rho)}{u\Sigma} \times[\{(\alpha-5)\sinh u+2u\cosh u\}\cosh u\zeta-2u\zeta\sinh u\sinh u\zeta]\,du. \tag{4·2·48}$$

It will be necessary to represent these displacements and stresses by Fourier expansions in the co-ordinate ζ and the required results may be obtained by using the expansions

$$\cosh u\zeta = u\sinh u\left\{\frac{1}{u^2}+2\sum_{r=1}^{\infty}\frac{(-)^r\cos r\pi\zeta}{u^2+r^2\pi^2}\right\} \quad (-1\leqslant\zeta\leqslant 1), \tag{4·2·49}$$

$$\zeta\sinh u\zeta = \frac{\cosh u}{u}-\frac{\sinh u}{u^2} +2\sum_{r=1}^{\infty}\left\{\frac{u\cosh u}{u^2+r^2\pi^2}-\frac{(u^2-r^2\pi^2)\sinh u}{(u^2+r^2\pi^2)^2}\right\}(-)^r\cos r\pi\zeta \quad (-1\leqslant\zeta\leqslant 1), \tag{4·2·50}$$

$$\sinh u\zeta = 2\sinh u\sum_{r=1}^{\infty}\frac{(-)^{r+1}r\pi\sin r\pi\zeta}{u^2+r^2\pi^2} \quad (-1<\zeta<1), \tag{4·2·51}$$

$$\zeta\cosh u\zeta = 2\sum_{r=1}^{\infty}\left\{\frac{\cosh u}{u^2+r^2\pi^2}-\frac{2u\sinh u}{(u^2+r^2\pi^2)^2}\right\}(-)^{r+1}r\pi\sin r\pi\zeta \quad (-1<\zeta<1). \tag{4·2·52}$$

Term-by-term integration of the various Fourier series presents no difficulty so that, with the help of the expressions (4·2·49) to (4·2·52), the complete displacements and stresses which satisfy the boundary condition (2·1·1) can be put in the forms

$$\left.\begin{aligned} u_r &= \left\{\alpha_0'+\tfrac{1}{2}\,{}^0\alpha_0+\sum_{r=1}^{\infty}{}^0\alpha_r\cos r\pi\zeta\right\}\cos n\theta,\\ u_\theta &= \left\{\beta_0'+\tfrac{1}{2}\,{}^0\beta_0+\sum_{r=1}^{\infty}{}^0\beta_r\cos r\pi\zeta\right\}\sin n\theta,\\ u_z &= \left\{\sum_{r=1}^{\infty}{}^0\gamma_r\sin r\pi\zeta\right\}\cos n\theta,\end{aligned}\right\} \tag{4·2·53}$$

$$\left.\begin{aligned} \widehat{rr} &= 2\mu\left\{a_0'+\tfrac{1}{2}\,{}^0a_0+\sum_{r=1}^{\infty}{}^0a_r\cos r\pi\zeta\right\}\cos n\theta,\\ \widehat{r\theta} &= 2\mu\left\{b_0'+\tfrac{1}{2}\,{}^0b_0+\sum_{r=1}^{\infty}{}^0b_r\cos r\pi\zeta\right\}\sin n\theta,\\ \widehat{rz} &= 2\mu\left\{\sum_{r=1}^{\infty}{}^0c_r\sin r\pi\zeta\right\}\cos n\theta,\\ \widehat{\theta z} &= 2\mu\left\{\sum_{r=1}^{\infty}{}^0d_r\sin r\pi\zeta\right\}\sin n\theta,\\ \widehat{zz} &= 2\mu\left\{e_0'+\tfrac{1}{2}\,{}^0e_0+\sum_{r=1}^{\infty}{}^0e_r\cos r\pi\zeta\right\}\cos n\theta,\\ \widehat{\theta\theta} &= 2\mu\left\{f_0'+\tfrac{1}{2}\,{}^0f_0+\sum_{r=1}^{\infty}{}^0f_r\cos r\pi\zeta\right\}\cos n\theta,\end{aligned}\right\} \tag{4·2·54}$$

where

$$\left.\begin{aligned} \alpha_0' &= \frac{2(\alpha+n-1)\,a\lambda^{n-1}}{(n-1)\,\rho^{n-1}} \quad (n\geqslant 2),\\ &= -2a(2\alpha\log\rho+\alpha-1) \quad (n=1),\\ \beta_0' &= \frac{2(-\alpha+n-1)\,a\lambda^{n-1}}{(n-1)\,\rho^{n-1}} \quad (n\geqslant 2),\\ &= 2a(2\alpha\log\rho+\alpha+1) \quad (n=1),\end{aligned}\right\} \tag{4·2·55}$$

$$\left.\begin{aligned} a_0' &= -2(n+2)\,\lambda^n/\rho^n \quad (n \geqslant 2), \\ &= -2(\alpha+3)\,\lambda/\rho \quad (n=1), \\ b_0' &= -2n\lambda^n/\rho^n \quad (n \geqslant 2), \\ &= 2(\alpha-1)\,\lambda/\rho \quad (n=1), \\ a_0'+f_0' &= -8\lambda^n/\rho^n \quad (n \geqslant 1), \\ e_0' &= -8\eta\lambda^n/\rho^n \quad (n \geqslant 1), \end{aligned}\right\} \tag{4·2·56}$$

and for $r \geqslant 0$,

$$^0\alpha_r = 128\eta nh(-)^r \int_0^\infty J_n(u\lambda)\, J_n'(u\rho) \sinh^2 u \left\{\frac{\eta}{u^2+r^2\pi^2} - \frac{r^2\pi^2}{(u^2+r^2\pi^2)^2}\right\} \frac{du}{u\Sigma}, \tag{4·2·57}$$

$$^0\beta_r = \frac{128\eta n^2 h(-)^{r+1}}{\rho} \int_0^\infty J_n(u\lambda)\, J_n(u\rho) \sinh^2 u \left\{\frac{\eta}{u^2+r^2\pi^2} - \frac{r^2\pi^2}{(u^2+r^2\pi^2)^2}\right\} \frac{du}{u^2\Sigma}, \tag{4·2·58}$$

$$^0\gamma_r = 128\eta nr\pi h(-)^{r+1} \int_0^\infty J_n(u\lambda)\, J_n(u\rho) \sinh^2 u \left\{\frac{2-\eta}{u^2+r^2\pi^2} - \frac{r^2\pi^2}{(u^2+r^2\pi^2)^2}\right\} \frac{du}{u^2\Sigma}, \tag{4·2·59}$$

$$^0a_r = 128\eta n(-)^r \int_0^\infty J_n(u\lambda) \sinh^2 u \left\{\frac{\eta\{J_n(u\rho)+J_n''(u\rho)\}}{u^2+r^2\pi^2} - \frac{r^2\pi^2 J_n''(u\rho)}{(u^2+r^2\pi^2)^2}\right\} \frac{du}{\Sigma}, \tag{4·2·60}$$

$$^0b_r = \frac{128\eta n^2(-)^r}{\rho^2} \int_0^\infty J_n(u\lambda)\, \{J_n(u\rho) - u\rho J_n'(u\rho)\} \sinh^2 u \left\{\frac{\eta}{u^2+r^2\pi^2} - \frac{r^2\pi^2}{(u^2+r^2\pi^2)^2}\right\} \frac{du}{u^2\Sigma}, \tag{4·2·61}$$

$$^0c_r = 128\eta nr\pi(-)^{r+1} \int_0^\infty \frac{J_n(u\lambda)\, J_n'(u\rho)\, u \sinh^2 u}{(u^2+r^2\pi^2)^2\,\Sigma}\, du, \tag{4·2·62}$$

$$^0d_r = \frac{128\eta n^2 r\pi(-)^r}{\rho} \int_0^\infty \frac{J_n(u\lambda)\, J_n(u\rho) \sinh^2 u}{(u^2+r^2\pi^2)^2\,\Sigma}\, du, \tag{4·2·63}$$

$$^0e_r = 128\eta n(-)^r \int_0^\infty \frac{J_n(u\lambda)\, J_n(u\rho)\, u^2 \sinh^2 u}{(u^2+r^2\pi^2)^2\,\Sigma}\, du, \tag{4·2·64}$$

$$^0a_r + {}^0f_r = 128\eta n(-)^r \int_0^\infty J_n(u\lambda)\, J_n(u\rho) \sinh^2 u \left\{\frac{\eta}{u^2+r^2\pi^2} + \frac{r^2\pi^2}{(u^2+r^2\pi^2)^2}\right\} \frac{du}{\Sigma}. \tag{4·2·65}$$

4·3. In this and the next two sections ∞^3 solutions of the elastic equations of equilibrium are found which give zero stresses at infinity and which satisfy the boundary conditions (2·1·1). The functions ψ, ω, ϕ and χ in the standard solutions A to D are harmonic functions and the harmonic functions which are needed here are of the forms

$$\left.\begin{aligned} &K_n(m\pi\rho) \cos m\pi\zeta \begin{matrix}\cos n\theta \\ \sin n\theta\end{matrix}, \\ &J_n(u\rho) \cosh u\zeta \cos n\theta. \end{aligned}\right\} \tag{4·3·1}$$

Consider first the solutions of type A which are derived from the potential function

$$\psi_m = a^2 I_n(m\pi\lambda)\, K_n(m\pi\rho) \cos m\pi\zeta \sin n\theta, \tag{4·3·2}$$

where m, n take all integral values, the constant factor $I_n(m\pi\lambda)$ being included for convenience. $I_n(m\pi\lambda)$ and $K_n(m\pi\rho)$ are the modified Bessel functions of order n of the first and second kind respectively. The corresponding displacements and stresses which are found from (3·2·2) and (3·2·3) are

$$\left.\begin{aligned} &u_r = \mu_m \cos m\pi\zeta \cos n\theta, && u_\theta = \nu_m \cos m\pi\zeta \sin n\theta,\\ &\widehat{rr} = 2\mu g_m \cos m\pi\zeta \cos n\theta, && \widehat{r\theta} = 2\mu h_m \cos m\pi\zeta \sin n\theta,\\ &\widehat{rz} = 2\mu i_m \sin m\pi\zeta \cos n\theta, && \widehat{\theta z} = 2\mu j_m \sin m\pi\zeta \sin n\theta,\\ &\widehat{\theta\theta} = 2\mu k_m \cos m\pi\zeta \cos n\theta, && \widehat{zz} = 0, \quad u_z = 0, \end{aligned}\right\} \tag{4·3·3}$$

where

$$\left.\begin{aligned} \mu_m &= 2an(\lambda/\rho)\, I_n(m\pi\lambda)\, K_n(m\pi\rho),\\ \nu_m &= -2m\pi a\lambda\, I_n(m\pi\lambda)\, K_n'(m\pi\rho),\\ g_m &= 2n(\lambda/\rho)^2\, I_n(m\pi\lambda)\, \{m\pi\rho K_n'(m\pi\rho) - K_n(m\pi\rho)\},\\ h_m &= -(\lambda/\rho)^2\, I_n(m\pi\lambda)\, \{m^2\pi^2\rho^2 K_n''(m\pi\rho) - m\pi\rho K_n'(m\pi\rho) + n^2 K_n(m\pi\rho)\},\\ i_m &= -nm\pi(\lambda^2/\rho)\, I_n(m\pi\lambda)\, K_n(m\pi\rho),\\ j_m &= m^2\pi^2\lambda^2 I_n(m\pi\lambda)\, K_n'(m\pi\rho),\\ k_m &+ g_m = 0. \end{aligned}\right\} \tag{4·3·4}$$

The system of stresses (4·3·3) satisfies the boundary conditions (2·1·1) when $\zeta = \pm 1$, without modification.

4·4. The next set of displacements and stresses are derived from a solution of type B of the elastic equations by using ω in the form

$$\omega = (h^2/m^2\pi^2)\, I_n(m\pi\lambda)\, K_n(m\pi\rho) \cos m\pi\zeta \cos n\theta, \tag{4·4·1}$$

and by substituting this function in equations (3·2·4) and (3·2·5). In particular, when $\zeta = \pm 1$, the shear stresses $\widehat{rz}$, $\widehat{\theta z}$ vanish and

$$\widehat{zz} = 2\mu(-)^{m+1}\, I_n(m\pi\lambda)\, K_n(m\pi\rho) \cos n\theta. \tag{4·4·2}$$

In order to satisfy the boundary conditions (2·1·1) when $\zeta = \pm 1$ it is now necessary to find a stress system for the region $-1 \leqslant \zeta \leqslant 1$, $\rho \geqslant \lambda$, of the plate which is zero at infinity and which cancels the stress $\widehat{zz}$ on the faces of the plate without introducing shear stresses $\widehat{rz}$ and $\widehat{\theta z}$. The solutions of types B and C which are given in (4·2·28) are again suitable for this purpose and the conditions at $\zeta = \pm 1$ now give

$$\left.\begin{aligned} &\int_0^\infty u\{f(u) \sinh u + g(u)\, (\sinh u + 2u \cosh u)\}\, J_n(u\rho)\, du = 0,\\ &\int_0^\infty u^2\{f(u) \cosh u - g(u)\, (\cosh u - 2u \sinh u)\}\, J_n(u\rho)\, du = (-)^m\, h^2 I_n(m\pi\lambda) K_n(m\pi\rho) \quad (\rho \geqslant \lambda). \end{aligned}\right\} \tag{4·4·3}$$

The solution of these integral equations may be obtained by using the result (Watson 1944, p. 429)

$$\int_0^\infty \frac{J_n(u\lambda)\, J_n(u\rho)\, u}{u^2 + m^2\pi^2}\, du = I_n(m\pi\lambda)\, K_n(m\pi\rho) \quad (\rho \geqslant \lambda). \tag{4·4·4}$$

Thus

$$\left.\begin{aligned} &f(u) \sinh u + g(u)\, (\sinh u + 2u \cosh u) = 0,\\ &f(u) \cosh u - g(u)\, (\cosh u - 2u \sinh u) = \frac{(-)^m\, h^2 J_n(u\lambda)}{u(u^2 + m^2\pi^2)}, \end{aligned}\right\} \tag{4·4·5}$$

and therefore

$$\left.\begin{aligned} f(u) &= \frac{(-)^m h^2(\sinh u + 2u\cosh u)\, J_n(u\lambda)}{u(u^2+m^2\pi^2)\,\Sigma},\\ g(u) &= \frac{(-)^{m+1} h^2 \sinh u\, J_n(u\lambda)}{u(u^2+m^2\pi^2)\,\Sigma}, \end{aligned}\right\} \tag{4·4·6}$$

where Σ is given by (4·2·34). The final forms for the potential functions ω, ϕ which together give stress systems which satisfy the boundary conditions (2·1·1) at the faces of the plate are, for all integral m and n,

$$\omega_m = (h^2/m^2\pi^2)\, I_n(m\pi\lambda)\, K_n(m\pi\rho)\cos m\pi\zeta \cos n\theta + (-)^m h^2 \cos n\theta \int_0^\infty \frac{(\sinh u + 2u\cosh u)\, J_n(u\lambda)\, J_n(u\rho)\cosh u\zeta}{u(u^2+m^2\pi^2)\,\Sigma}\,du, \tag{4·4·7}$$

$$\phi_m = (-)^{m+1} h^2 \cos n\theta \int_0^\infty \frac{\sinh u J_n(u\lambda)\, J_n(u\rho)\cosh u\zeta}{u(u^2+m^2\pi^2)\,\Sigma}\,du. \tag{4·4·8}$$

These integrals converge and may be differentiated the required number of times using the formulae (3·2·4) to (3·2·7) in order to obtain the displacements and stresses, which are as follows:

$$\frac{u_r}{\cos n\theta} = (h/m\pi)\, I_n(m\pi\lambda)\, K_n'(m\pi\rho)\cos m\pi\zeta + (-)^m h \int_0^\infty \frac{J_n(u\lambda)\, J_n'(u\rho)}{(u^2+m^2\pi^2)\,\Sigma} \times \{(\sinh u + 2u\cosh u)\cosh u\zeta - \sinh u(\alpha\cosh u\zeta + 2u\zeta\sinh u\zeta)\}\,du, \tag{4·4·9}$$

$$\frac{u_\theta}{\sin n\theta} = -(nh/\rho m^2\pi^2)\, I_n(m\pi\lambda)\, K_n(m\pi\rho)\cos m\pi\zeta + \frac{(-)^{m+1} nh}{\rho}\int_0^\infty \frac{J_n(u\lambda)\, J_n(u\rho)}{u(u^2+m^2\pi^2)\,\Sigma} \times \{(\sinh u + 2u\cosh u)\cosh u\zeta - \sinh u(\alpha\cosh u\zeta + 2u\zeta\sinh u\zeta)\}\,du, \tag{4·4·10}$$

$$\frac{u_z}{\cos n\theta} = -(h/m\pi)\, I_n(m\pi\lambda)\, K_n(m\pi\rho)\sin m\pi\zeta + (-)^m h \int_0^\infty \frac{J_n(u\lambda)\, J_n(u\rho)}{(u^2+m^2\pi^2)\,\Sigma} \times \{(\sinh u + 2u\cosh u)\sinh u\zeta + \sinh u(\alpha\sinh u\zeta - 2u\zeta\cosh u\zeta)\}\,du, \tag{4·4·11}$$

$$\frac{\widehat{rr}}{2\mu\cos n\theta} = I_n(m\pi\lambda)\, K_n''(m\pi\rho)\cos m\pi\zeta + (-)^m \int_0^\infty \frac{uJ_n(u\lambda)}{(u^2+m^2\pi^2)\,\Sigma} \times [\{\{(1-\alpha)\sinh u + 2u\cosh u\}\cosh u\zeta - 2u\zeta\sinh u\sinh u\zeta\}\, J_n''(u\rho) + (3-\alpha)\sinh u\cosh u\zeta\, J_n(u\rho)]\,du, \tag{4·4·12}$$

$$\frac{\widehat{r\theta}}{2\mu\sin n\theta} = (n/m^2\pi^2\rho^2)\, I_n(m\pi\lambda)\,\{K_n(m\pi\rho) - m\pi\rho K_n'(m\pi\rho)\}\cos m\pi\zeta + \frac{(-)^m n}{\rho^2}\int_0^\infty \frac{J_n(u\lambda)\,\{J_n(u\rho) - u\rho J_n'(u\rho)\}}{u(u^2+m^2\pi^2)\,\Sigma} \times [\{(1-\alpha)\sinh u + 2u\cosh u\}\cosh u\zeta - 2u\zeta\sinh u\sinh u\zeta]\,du, \tag{4·4·13}$$

$$\frac{\widehat{rz}}{2\mu\cos n\theta} = -I_n(m\pi\lambda)\, K_n'(m\pi\rho)\sin m\pi\zeta + 2(-)^m \int_0^\infty \frac{u^2 J_n(u\lambda)\, J_n'(u\rho)}{(u^2+m^2\pi^2)\,\Sigma}\{\cosh u\sinh u\zeta - \zeta\sinh u\cosh u\zeta\}\,du, \tag{4·4·14}$$

$$\frac{\widehat{\theta z}}{2\mu\sin n\theta} = (n/m\pi\rho)\, I_n(m\pi\lambda)\, K_n(m\pi\rho)\sin m\pi\zeta + \frac{2(-)^{m+1} n}{\rho}\int_0^\infty \frac{u J_n(u\lambda)\, J_n(u\rho)}{(u^2+m^2\pi^2)\,\Sigma}\{\cosh u\sinh u\zeta - \zeta\sinh u\cosh u\zeta\}\,du, \tag{4·4·15}$$

$$\frac{\widehat{zz}}{2\mu\cos n\theta} = -I_n(m\pi\lambda)\,K_n(m\pi\rho)\cos m\pi\zeta + 2(-)^m\int_0^\infty \frac{u\,J_n(u\lambda)\,J_n(u\rho)}{(u^2+m^2\pi^2)\,\Sigma}$$
$$\times\{(\sinh u + u\cosh u)\cosh u\zeta - u\zeta\sinh u\sinh u\zeta\}\,du, \quad (4\cdot4\cdot16)$$

$$\frac{\widehat{rr}+\widehat{\theta\theta}}{2\mu\cos n\theta} = I_n(m\pi\lambda)\,K_n(m\pi\rho)\cos m\pi\zeta + (-)^{m+1}\int_0^\infty \frac{u\,J_n(u\lambda)\,J_n(u\rho)}{(u^2+m^2\pi^2)\,\Sigma}$$
$$\times[\{(\alpha-5)\sinh u + 2u\cosh u\}\cosh u\zeta - 2u\zeta\sinh u\sinh u\zeta]\,du. \quad (4\cdot4\cdot17)$$

Fourier expansions for these displacements and stresses will be needed. With the help of (4·2·49) to (4·2·53), and observing that term-by-term integration of the resulting series is possible these expansions may be expressed in the form

$$\left.\begin{aligned} u_r &= \left\{\alpha'_m\cos m\pi\zeta + \tfrac{1}{2}\,{}^m\alpha_0 + \sum_{r=1}^\infty {}^m\alpha_r\cos r\pi\zeta\right\}\cos n\theta,\\ u_\theta &= \left\{\beta'_m\cos m\pi\zeta + \tfrac{1}{2}\,{}^m\beta_0 + \sum_{r=1}^\infty {}^m\beta_r\cos r\pi\zeta\right\}\sin n\theta,\\ u_z &= \left\{\gamma'_m\sin m\pi\zeta + \sum_{r=1}^\infty {}^m\gamma_r\sin r\pi\zeta\right\}\cos n\theta,\end{aligned}\right\} \quad (4\cdot4\cdot18)$$

$$\left.\begin{aligned} \widehat{rr} &= 2\mu\left\{a'_m\cos m\pi\zeta + \tfrac{1}{2}\,{}^m a_0 + \sum_{r=1}^\infty {}^m a_r\cos r\pi\zeta\right\}\cos n\theta,\\ \widehat{r\theta} &= 2\mu\left\{b'_m\cos m\pi\zeta + \tfrac{1}{2}\,{}^m b_0 + \sum_{r=1}^\infty {}^m b_r\cos r\pi\zeta\right\}\sin n\theta,\\ \widehat{rz} &= 2\mu\left\{c'_m\sin m\pi\zeta + \sum_{r=1}^\infty {}^m c_r\sin r\pi\zeta\right\}\cos n\theta,\\ \widehat{\theta z} &= 2\mu\left\{d'_m\sin m\pi\zeta + \sum_{r=1}^\infty {}^m d_r\sin r\pi\zeta\right\}\sin n\theta,\\ \widehat{zz} &= 2\mu\left\{e'_m\cos m\pi\zeta + \tfrac{1}{2}\,{}^m e_0 + \sum_{r=1}^\infty {}^m e_r\cos r\pi\zeta\right\}\cos n\theta,\\ \widehat{\theta\theta} &= 2\mu\left\{f'_m\cos m\pi\zeta + \tfrac{1}{2}\,{}^m f_0 + \sum_{r=1}^\infty {}^m f_r\cos r\pi\zeta\right\}\cos n\theta,\end{aligned}\right\} \quad (4\cdot4\cdot19)$$

where

$$\left.\begin{aligned} \alpha'_m &= (h/m\pi)\,I_n(m\pi\lambda)\,K'_n(m\pi\rho),\\ \beta'_m &= -(nh/\rho m^2\pi^2)\,I_n(m\pi\lambda)\,K_n(m\pi\rho),\\ \gamma'_m &= -(h/m\pi)\,I_n(m\pi\lambda)\,K_n(m\pi\rho),\\ a'_m &= I_n(m\pi\lambda)\,K''_n(m\pi\rho),\\ b'_m &= (n/m^2\pi^2\rho^2)\,I_n(m\pi\lambda)\,\{K_n(m\pi\rho) - m\pi\rho K'_n(m\pi\rho)\},\\ c'_m &= -I_n(m\pi\lambda)\,K'_n(m\pi\rho),\\ d'_m &= (n/m\pi\rho)\,I_n(m\pi\lambda)\,K_n(m\pi\rho),\\ e'_m &= -I_n(m\pi\lambda)\,K_n(m\pi\rho),\\ a'_m + f'_m &= I_n(m\pi\lambda)\,K_n(m\pi\rho),\end{aligned}\right\} \quad (4\cdot4\cdot20)$$

and
$${}^m\alpha_r = 8(-)^{m+r}h\int_0^\infty \frac{J_n(u\lambda)\,J'_n(u\rho)\,u\sinh^2 u}{(u^2+m^2\pi^2)\,\Sigma}\left\{\frac{\eta}{u^2+r^2\pi^2} - \frac{r^2\pi^2}{(u^2+r^2\pi^2)^2}\right\}du, \quad (4\cdot4\cdot21)$$

$${}^m\beta_r = \frac{8(-)^{m+r+1}nh}{\rho}\int_0^\infty \frac{J_n(u\lambda)\,J_n(u\rho)\sinh^2 u}{(u^2+m^2\pi^2)\,\Sigma}\left\{\frac{\eta}{u^2+r^2\pi^2} - \frac{r^2\pi^2}{(u^2+r^2\pi^2)^2}\right\}du, \quad (4\cdot4\cdot22)$$

$$ {}^m\gamma_r = 8(-)^{m+r+1} r\pi h \int_0^\infty \frac{J_n(u\lambda)\, J_n(u\rho) \sinh^2 u}{(u^2+m^2\pi^2)\, \Sigma} \left\{ \frac{2-\eta}{u^2+r^2\pi^2} - \frac{r^2\pi^2}{(u^2+r^2\pi^2)^2} \right\} du, \tag{4·4·23} $$

$$ {}^m a_r = 8(-)^{m+r} \int_0^\infty \frac{J_n(u\lambda)\, u^2 \sinh^2 u}{(u^2+m^2\pi^2)\, \Sigma} \left\{ \frac{\eta\{J_n(u\rho)+J_n''(u\rho)\}}{u^2+r^2\pi^2} - \frac{r^2\pi^2 J_n''(u\rho)}{(u^2+r^2\pi^2)^2} \right\} du, \tag{4·4·24} $$

$$ {}^m b_r = \frac{8(-)^{m+r}\, n}{\rho^2} \int_0^\infty \frac{J_n(u\lambda)\, \{J_n(u\rho) - u\rho J_n'(u\rho)\} \sinh^2 u}{(u^2+m^2\pi^2)\, \Sigma} \left\{ \frac{\eta}{u^2+r^2\pi^2} - \frac{r^2\pi^2}{(u^2+r^2\pi^2)^2} \right\} du, \tag{4·4·25} $$

$$ {}^m c_r = 8(-)^{m+r+1} r\pi \int_0^\infty \frac{J_n(u\lambda)\, J_n'(u\rho)\, u^3 \sinh^2 u}{(u^2+m^2\pi^2)\,(u^2+r^2\pi^2)^2\, \Sigma}\, du, \tag{4·4·26} $$

$$ {}^m d_r = \frac{8(-)^{m+r}\, n r\pi}{\rho} \int_0^\infty \frac{J_n(u\lambda)\, J_n(u\rho)\, u^2 \sinh^2 u}{(u^2+m^2\pi^2)\,(u^2+r^2\pi^2)^2\, \Sigma}\, du, \tag{4·4·27} $$

$$ {}^m e_r = 8(-)^{m+r} \int_0^\infty \frac{J_n(u\lambda)\, J_n(u\rho)\, u^4 \sinh^2 u}{(u^2+m^2\pi^2)\,(u^2+r^2\pi^2)^2\, \Sigma}\, du, \tag{4·4·28} $$

$$ {}^m a_r + {}^m f_r = 8(-)^{m+r} \int_0^\infty \frac{J_n(u\lambda)\, J_n(u\rho)\, u^2 \sinh^2 u}{(u^2+m^2\pi^2)\, \Sigma} \left\{ \frac{\eta}{u^2+r^2\pi^2} + \frac{r^2\pi^2}{(u^2+r^2\pi^2)^2} \right\} du. \tag{4·4·29} $$

4·5. The third group of displacements and stresses which satisfy the boundary conditions (2·1·1) could be found in a similar manner to that used in § 4·4 by starting with a suitable expression for ϕ (solution C) and then adding other functions of the type given in (4·2·28). It is convenient, however, to begin with solutions of the types B and D of the form

$$ \left.\begin{aligned} \omega' &= -2(\lambda h^2/m\pi)\, I_n'(m\pi\lambda)\, K_n(m\pi\rho) \cos m\pi\zeta \cos n\theta, \\ \chi &= \quad (h^2/m^2\pi^2)\, I_n(m\pi\lambda)\, K_n(m\pi\rho) \cos m\pi\zeta \cos n\theta. \end{aligned}\right\} \tag{4·5·1} $$

The corresponding displacements and stresses are found from (3·2·4), (3·2·5), (3·2·8) and (3·2·9). In particular, when $\zeta = \pm 1$, it is found that the shear stresses $\widehat{rz}$, $\widehat{\theta z}$ vanish and the normal stress $\widehat{zz}$ is

$$ \left.\begin{aligned} \widehat{zz} &= 4\mu(-)^m\, m\pi \{\lambda I_n'(m\pi\lambda)\, K_n(m\pi\rho) + \rho I_n(m\pi\lambda)\, K_n'(m\pi\rho)\} \cos n\theta, \\ &= 4\mu(-)^m\, m\pi \frac{d}{dm\pi} \{I_n(m\pi\lambda)\, K_n(m\pi\rho)\} \cos n\theta. \end{aligned}\right\} \tag{4·5·2} $$

Solutions of the types B and C which are given in (4·2·28) and which give zero stress at infinity are now added so that they cancel this value of $\widehat{zz}$ on the faces $\zeta = \pm 1$ of the plate without introducing shear stresses $\widehat{rz}$, $\widehat{\theta z}$ on these faces. Hence

$$ \left.\begin{aligned} &\int_0^\infty u\{f(u) \sinh u + g(u)\,(\sinh u + 2u \cosh u)\}\, J_n(u\rho)\, du = 0, \\ &\int_0^\infty u^2\{f(u) \cosh u - g(u)\,(\cosh u - 2u \sinh u)\}\, J_n(u\rho)\, du \\ &\qquad\qquad = 2(-)^{m+1}\, m\pi h^2 \frac{d}{dm\pi} I_n(m\pi\lambda)\, K_n(m\pi\rho) \quad (\rho \geqslant \lambda), \end{aligned}\right\} \tag{4·5·3} $$

and the solution of these integral equations can be found with the help of the equation (Watson 1944, p. 429)

$$ \int_0^\infty \frac{J_n(u\lambda)\, J_n(u\rho)\, u}{(u^2+m^2\pi^2)^2}\, du = -\frac{1}{2m\pi} \frac{d}{dm\pi} I_n(m\pi\lambda)\, K_n(m\pi\rho) \quad (\rho \geqslant \lambda). \tag{4·5·4} $$

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From (4·5·3) and (4·5·4)

$$\left.\begin{aligned} f(u)\sinh u+g(u)(\sinh u+2u\cosh u) &= 0,\\ f(u)\cosh u-g(u)(\cosh u-2u\sinh u) &= \frac{4(-)^m m^2\pi^2h^2 J_n(u\lambda)}{u(u^2+m^2\pi^2)^2},\end{aligned}\right\} \tag{4·5·5}$$

and solving for $f(u)$ and $g(u)$ gives

$$\left.\begin{aligned} f(u) &= \frac{4(-)^m m^2\pi^2h^2(\sinh u+2u\cosh u)J_n(u\lambda)}{u(u^2+m^2\pi^2)^2\Sigma},\\ g(u) &= \frac{4(-)^{m+1} m^2\pi^2h^2\sinh u\, J_n(u\lambda)}{u(u^2+m^2\pi^2)^2\Sigma},\end{aligned}\right\} \tag{4·5·6}$$

where Σ is defined by (4·2·32). The final forms for the potential functions ω'_m, ϕ'_m, χ_m of the types B, C, D respectively, which together give stresses satisfying the boundary conditions (2·1·1) at the faces of the plate are, for all integral values of m and n,

$$\begin{aligned}\omega'_m = &-2(\lambda h^2/m\pi)I'_n(m\pi\lambda)K_n(m\pi\rho)\cos m\pi\zeta\cos n\theta\\ &+4(-)^m m^2\pi^2h^2\cos n\theta\int_0^\infty\frac{(\sinh u+2u\cosh u)J_n(u\lambda)J_n(u\rho)\cosh u\zeta}{u(u^2+m^2\pi^2)^2\Sigma}du,\end{aligned} \tag{4·5·7}$$

$$\phi'_m = 4(-)^{m+1}m^2\pi^2h^2\cos n\theta\int_0^\infty\frac{\sinh u\, J_n(u\lambda)J_n(u\rho)\cosh u\zeta}{u(u^2+m^2\pi^2)^2\Sigma}du, \tag{4·5·8}$$

$$\chi_m = (h^2/m^2\pi^2)I_n(m\pi\lambda)K_n(m\pi\rho)\cos m\pi\zeta\cos n\theta. \tag{4·5·9}$$

The integrals in (4·5·7) and (4·5·8) converge and may be differentiated the required number of times using the formulae (3·2·4) to (3·2·7) in order to obtain the displacements and stresses. From (4·5·7) to (4·5·9) and (3·2·4) to (3·2·9) the displacements and stresses are found to be

$$\begin{aligned}\frac{u_r}{\cos n\theta} = &(h/m\pi)\{-2m\pi\rho I_n(m\pi\lambda)K_n(m\pi\rho)-2m\pi\lambda I'_n(m\pi\lambda)K'_n(m\pi\rho)\\ &+(\alpha+5)I_n(m\pi\lambda)K'_n(m\pi\rho)\}\cos m\pi\zeta+4(-)^m m^2\pi^2h\int_0^\infty\frac{J_n(u\lambda)J'_n(u\rho)}{(u^2+m^2\pi^2)^2\Sigma}\\ &\times\{(\sinh u+2u\cosh u)\cosh u\zeta-\sinh u(\alpha\cosh u\zeta+2u\zeta\sinh u\zeta)\}du,\end{aligned} \tag{4·5·10}$$

$$\begin{aligned}\frac{u_\theta}{\sin n\theta} = &(nh/\rho m^2\pi^2)\{-(\alpha+5)I_n(m\pi\lambda)K_n(m\pi\rho)+2m\pi\lambda I'_n(m\pi\lambda)K_n(m\pi\rho)\}\cos m\pi\zeta\\ &+\frac{4(-)^{m+1}nhm^2\pi^2}{\rho}\int_0^\infty\frac{J_n(u\lambda)J_n(u\rho)}{u(u^2+m^2\pi^2)^2\Sigma}\{(\sinh u+2u\cosh u)\cosh u\zeta\\ &-\sinh u(\alpha\cosh u\zeta+2u\zeta\sinh u\zeta)\}du,\end{aligned} \tag{4·5·11}$$

$$\begin{aligned}\frac{u_z}{\cos n\theta} = &(h/m\pi)\{2m\pi\rho I_n(m\pi\lambda)K'_n(m\pi\rho)+2m\pi\lambda I'_n(m\pi\lambda)K_n(m\pi\rho)\\ &+(\alpha-3)I_n(m\pi\lambda)K_n(m\pi\rho)\}\sin m\pi\zeta+4(-)^m m^2\pi^2h\int_0^\infty\frac{J_n(u\lambda)J_n(u\rho)}{(u^2+m^2\pi^2)^2\Sigma}\\ &\times\{(\sinh u+2u\cosh u)\sinh u\zeta+\sinh u(\alpha\sinh u\zeta-2u\zeta\cosh u\zeta)\}du,\end{aligned} \tag{4·5·12}$$

$$\begin{aligned}\frac{\widehat{rr}}{2\mu\cos n\theta} = &[\{(\alpha+5)K''_n(m\pi\rho)-2m\pi\rho K'_n(m\pi\rho)-(\alpha-1)K_n(m\pi\rho)\}I_n(m\pi\lambda)\\ &-2m\pi\lambda I'_n(m\pi\lambda)K''_n(m\pi\rho)]\cos m\pi\zeta+4(-)^m m^2\pi^2\int_0^\infty\frac{uJ_n(u\lambda)}{(u^2+m^2\pi^2)^2\Sigma}\\ &\times[\{\{(1-\alpha)\sinh u+2u\cosh u\}\cosh u\zeta-2u\zeta\sinh u\sinh u\zeta\}J''_n(u\rho)\\ &+(3-\alpha)\sinh u\cosh u\zeta J_n(u\rho)]du,\end{aligned} \tag{4·5·13}$$

$$\frac{\widehat{r\theta}}{2\mu\sin n\theta} = \frac{n}{m^2\pi^2\rho^2}[\{\{(\alpha+5)\{K_n(m\pi\rho)-m\pi\rho K_n'(m\pi\rho)\}+m^2\pi^2\rho^2K_n(m\pi\rho)\}I_n(m\pi\lambda)$$
$$-2m\pi\lambda\{K_n(m\pi\rho)-m\pi\rho K_n'(m\pi\rho)\}I_n'(m\pi\lambda)]\cos m\pi\zeta$$
$$+\frac{4(-)^m nm^2\pi^2}{\rho^2}\int_0^\infty \frac{J_n(u\lambda)\{J_n(u\rho)-u\rho J_n'(u\rho)\}}{u(u^2+m^2\pi^2)^2\Sigma}$$
$$\times[\{(1-\alpha)\sinh u+2u\cosh u\}\cosh u\zeta-2u\zeta\sinh u\sinh u\zeta]\,du, \quad (4\cdot5\cdot14)$$

$$\frac{\widehat{rz}}{2\mu\cos n\theta} = [\{m\pi\rho K_n''(m\pi\rho)+m\pi\rho K_n(m\pi\rho)-3K_n'(m\pi\rho)\}I_n(m\pi\lambda)$$
$$+2m\pi\lambda I_n'(m\pi\lambda)K_n'(m\pi\rho)]\sin m\pi\zeta+8(-)^m m^2\pi^2\int_0^\infty \frac{u^2J_n(u\lambda)J_n'(u\rho)}{(u^2+m^2\pi^2)^2\Sigma}$$
$$\times\{\cosh u\sinh u\zeta-\zeta\sinh u\cosh u\zeta\}\,du, \quad (4\cdot5\cdot15)$$

$$\frac{\widehat{\theta z}}{2\mu\sin n\theta} = \frac{n}{m\pi\rho}[\{4K_n(m\pi\rho)-m\pi\rho K_n'(m\pi\rho)\}I_n(m\pi\lambda)-2m\pi\lambda I_n'(m\pi\lambda)K_n(m\pi\rho)]\sin m\pi\zeta$$
$$+\frac{8(-)^{m+1}nm^2\pi^2}{\rho}\int_0^\infty \frac{uJ_n(u\lambda)J_n(u\rho)}{(u^2+m^2\pi^2)^2\Sigma}\{\cosh u\sinh u\zeta-\zeta\sinh u\cosh u\zeta\}\,du, \quad (4\cdot5\cdot16)$$

$$\frac{\widehat{zz}}{2\mu\cos n\theta} = 2m\pi\{\lambda I_n'(m\pi\lambda)K_n(m\pi\rho)+\rho I_n(m\pi\lambda)K_n'(m\pi\rho)\}\cos m\pi\zeta$$
$$+8(-)^m m^2\pi^2\int_0^\infty \frac{uJ_n(u\lambda)J_n(u\rho)}{(u^2+m^2\pi^2)^2\Sigma}\{(\sinh u+u\cosh u)\cosh u\zeta-u\zeta\sinh u\sinh u\zeta\}\,du,$$
$$(4\cdot5\cdot17)$$

$$\frac{\widehat{rr}+\widehat{\theta\theta}}{2\mu\cos n\theta} = [-2m\pi\{\lambda I_n'(m\pi\lambda)K_n(m\pi\rho)+\rho I_n(m\pi\lambda)K_n'(m\pi\rho)\}$$
$$+(7-\alpha)I_n(m\pi\lambda)K_n(m\pi\rho)]\cos m\pi\zeta+4(-)^{m+1}m^2\pi^2\int_0^\infty \frac{uJ_n(u\lambda)J_n(u\rho)}{(u^2+m^2\pi^2)^2\Sigma}$$
$$\times[\{(\alpha-5)\sinh u+2u\cosh u\}\cosh u\zeta-2u\zeta\sinh u\sinh u\zeta]\,du. \quad (4\cdot5\cdot18)$$

These displacements and stresses can be expanded in Fourier series. Thus, using (4·2·49) to (4·2·53), and noting that term by term integration of the resulting series is possible,

$$\left.\begin{aligned} u_r &= \left\{\xi_m'\cos m\pi\zeta+\tfrac{1}{2}\,{}^m\xi_0+\sum_{r=1}^\infty {}^m\xi_r\cos r\pi\zeta\right\}\cos n\theta,\\ u_\theta &= \left\{\eta_m'\cos m\pi\zeta+\tfrac{1}{2}\,{}^m\eta_0+\sum_{r=1}^\infty {}^m\eta_r\cos r\pi\zeta\right\}\sin n\theta,\\ u_z &= \left\{\tau_m'\sin m\pi\zeta+\sum_{r=1}^\infty {}^m\tau_r\sin r\pi\zeta\right\}\cos n\theta,\end{aligned}\right\} \quad (4\cdot5\cdot19)$$

$$\left.\begin{aligned} \widehat{rr} &= 2\mu\left\{u_m'\cos m\pi\zeta+\tfrac{1}{2}\,{}^m u_0+\sum_{r=1}^\infty {}^m u_r\cos r\pi\zeta\right\}\cos n\theta,\\ \widehat{r\theta} &= 2\mu\left\{v_m'\cos m\pi\zeta+\tfrac{1}{2}\,{}^m v_0+\sum_{r=1}^\infty {}^m v_r\cos r\pi\zeta\right\}\sin n\theta,\\ \widehat{rz} &= 2\mu\left\{w_m'\sin m\pi\zeta+\sum_{r=1}^\infty {}^m w_r\sin r\pi\zeta\right\}\cos n\theta,\\ \widehat{\theta z} &= 2\mu\left\{x_m'\sin m\pi\zeta+\sum_{r=1}^\infty {}^m x_r\sin r\pi\zeta\right\}\sin n\theta,\\ \widehat{zz} &= 2\mu\left\{y_m'\cos m\pi\zeta+\tfrac{1}{2}\,{}^m y_0+\sum_{r=1}^\infty {}^m y_r\cos r\pi\zeta\right\}\cos n\theta,\\ \widehat{\theta\theta} &= 2\mu\left\{z_m'\cos m\pi\zeta+\tfrac{1}{2}\,{}^m z_0+\sum_{r=1}^\infty {}^m z_r\cos r\pi\zeta\right\}\cos n\theta,\end{aligned}\right\} \quad (4\cdot5\cdot20)$$

where

$$\left.\begin{aligned}\xi'_m &= (h/m\pi)\{-2m\pi\rho\, I_n(m\pi\lambda)\, K_n(m\pi\rho) - 2m\pi\lambda\, I'_n(m\pi\lambda)\, K'_n(m\pi\rho) + (\alpha+5)\, I_n(m\pi\lambda)\, K'_n(m\pi\rho)\},\\ \eta'_m &= (nh/\rho m^2\pi^2)\{2m\pi\lambda\, I'_n(m\pi\lambda)\, K_n(m\pi\rho) - (\alpha+5)\, I_n(m\pi\lambda)\, K_n(m\pi\rho)\},\\ \tau'_m &= (h/m\pi)\{2m\pi\rho\, I_n(m\pi\lambda)\, K'_n(m\pi\rho) + 2m\pi\lambda\, I'_n(m\pi\lambda)\, K_n(m\pi\rho) + (\alpha-3)\, I_n(m\pi\lambda)\, K_n(m\pi\rho)\},\end{aligned}\right\} \tag{4·5·21}$$

$$\left.\begin{aligned}u'_m &= \{(\alpha+5)\, K''_n(m\pi\rho) - 2m\pi\rho\, K'_n(m\pi\rho) - (\alpha-1)\, K_n(m\pi\rho)\}\, I_n(m\pi\lambda)\\ &\qquad - 2m\pi\lambda I'_n(m\pi\lambda)\, K''_n(m\pi\rho),\\ v'_m &= \frac{n}{m^2\pi^2\rho^2}[\{(\alpha+5)\{K_n(m\pi\rho) - m\pi\rho\, K'_n(m\pi\rho)\} + m^2\pi^2\rho^2\, K_n(m\pi\rho)\}\, I_n(m\pi\lambda)\\ &\qquad - 2m\pi\lambda\{K_n(m\pi\rho) - m\pi\rho\, K'_n(m\pi\rho)\}\, I'_n(m\pi\lambda)],\\ w'_m &= \{m\pi\rho\, K''_n(m\pi\rho) + m\pi\rho\, K_n(m\pi\rho) - 3K'_n(m\pi\rho)\}\, I_n(m\pi\lambda)\\ &\qquad + 2m\pi\lambda\, I'_n(m\pi\lambda)\, K'_n(m\pi\rho),\\ x'_m &= \frac{n}{m\pi\rho}[\{4K_n(m\pi\rho) - m\pi\rho\, K'_n(m\pi\rho)\}\, I_n(m\pi\lambda) - 2m\pi\lambda\, I'_n(m\pi\lambda)\, K_n(m\pi\rho)],\\ y'_m &= 2m\pi\{\lambda I'_n(m\pi\lambda)\, K_n(m\pi\rho) + \rho I_n(m\pi\lambda)\, K'_n(m\pi\rho)\},\\ u'_m + z'_m &= -2m\pi\{\lambda I'_n(m\pi\lambda)\, K_n(m\pi\rho) + \rho I_n(m\pi\lambda)\, K'_n(m\pi\rho)\} + (7-\alpha)\, I_n(m\pi\lambda)\, K_n(m\pi\rho),\end{aligned}\right\} \tag{4·5·22}$$

$${}^m\xi_r = 32(-)^{m+r}\, hm^2\pi^2 \int_0^\infty \frac{J_n(u\lambda)\, J'_n(u\rho)\, u \sinh^2 u}{(u^2+m^2\pi^2)^2\,\Sigma}\left\{\frac{\eta}{u^2+r^2\pi^2} - \frac{r^2\pi^2}{(u^2+r^2\pi^2)^2}\right\} du, \tag{4·5·23}$$

$${}^m\eta_r = \frac{32(-)^{m+r+1} nhm^2\pi^2}{\rho} \int_0^\infty \frac{J_n(u\lambda)\, J_n(u\rho) \sinh^2 u}{(u^2+m^2\pi^2)^2\,\Sigma}\left\{\frac{\eta}{u^2+r^2\pi^2} - \frac{r^2\pi^2}{(u^2+r^2\pi^2)^2}\right\} du, \tag{4·5·24}$$

$${}^m\tau_r = 32(-)^{m+r+1}\, rhm^2\pi^3 \int_0^\infty \frac{J_n(u\lambda)\, J_n(u\rho) \sinh^2 u}{(u^2+m^2\pi^2)^2\,\Sigma}\left\{\frac{2-\eta}{u^2+r^2\pi^2} - \frac{r^2\pi^2}{(u^2+r^2\pi^2)^2}\right\} du, \tag{4·5·25}$$

$${}^m u_r = 32(-)^{m+r}\, m^2\pi^2 \int_0^\infty \frac{J_n(u\lambda)\, u^2 \sinh^2 u}{(u^2+m^2\pi^2)^2\,\Sigma}\left\{\frac{\eta\{J_n(u\rho) + J''_n(u\rho)\}}{u^2+r^2\pi^2} - \frac{r^2\pi^2 J''_n(u\rho)}{(u^2+r^2\pi^2)^2}\right\} du, \tag{4·5·26}$$

$${}^m v_r = \frac{32(-)^{m+r}\, nm^2\pi^2}{\rho^2} \int_0^\infty \frac{J_n(u\lambda)\{J_n(u\rho) - u\rho J'_n(u\rho)\} \sinh^2 u}{(u^2+m^2\pi^2)^2\,\Sigma}\left\{\frac{\eta}{u^2+r^2\pi^2} - \frac{r^2\pi^2}{(u^2+r^2\pi^2)^2}\right\} du, \tag{4·5·27}$$

$${}^m w_r = 32(-)^{m+r+1}\, rm^2\pi^3 \int_0^\infty \frac{J_n(u\lambda)\, J'_n(u\rho)\, u^3 \sinh^2 u}{(u^2+m^2\pi^2)^2\,(u^2+r^2\pi^2)^2\,\Sigma}\, du, \tag{4·5·28}$$

$${}^m x_r = \frac{32(-)^{m+r}\, nrm^2\pi^3}{\rho} \int_0^\infty \frac{J_n(u\lambda)\, J_n(u\rho)\, u^2 \sinh^2 u}{(u^2+m^2\pi^2)^2\,(u^2+r^2\pi^2)^2\,\Sigma}\, du, \tag{4·5·29}$$

$${}^m y_r = 32(-)^{m+r}\, m^2\pi^2 \int_0^\infty \frac{J_n(u\lambda)\, J_n(u\rho)\, u^4 \sinh^2 u}{(u^2+m^2\pi^2)^2\,(u^2+r^2\pi^2)^2\,\Sigma}\, du, \tag{4·5·30}$$

$${}^m u_r + {}^m z_r = 32(-)^{m+r}\, m^2\pi^2 \int_0^\infty \frac{J_n(u\lambda)\, J_n(u\rho)\, u^2 \sinh^2 u}{(u^2+m^2\pi^2)^2\,\Sigma}\left\{\frac{\eta}{u^2+r^2\pi^2} + \frac{r^2\pi^2}{(u^2+r^2\pi^2)^2}\right\} du. \tag{4·5·31}$$

4·6. At first sight it would appear that ∞^3+4 independent stress systems have been found which satisfy the boundary conditions (2·1·1); ∞^3 in § 4·3 to § 4·5, one in § 4·1 and three in § 4·2. The three independent solutions (4·2·1), (4·2·4) and (4·2·7) (modified when necessary so as to satisfy the conditions (2·1·1)) were replaced by two others (4·2·19) and (4·2·35) to

(4·2·37). The third independent solution may then be taken to be (4·2·1) after a suitable function has been added in order to make the normal and shear stresses zero when $\zeta = \pm 1$. Restricting the argument to the cases $n \geq 2$, for convenience, it is found by the methods of the previous sections that the complete function ψ which is required, so that the boundary conditions (2·1·1) are satisfied, is

$$\psi = a^2\lambda^{n-2}\left\{\frac{\zeta^2}{\rho^n}+\frac{1}{2(n-1)\rho^{n-2}}\right\}\sin n\theta - \frac{4nh^2\lambda^n \sin n\theta}{\lambda'^n}\int_0^\infty \frac{J_n(u\lambda')\,J_n(u\rho)\cosh u\zeta}{u^2\sinh u}\,du \quad (\rho \geq \lambda > \lambda'), \tag{4·6·1}$$

the displacements and stresses then being given by (3·2·2) and (3·2·3). If $\cosh u\zeta$ is expanded in a Fourier series by (4·2·49) and if the series is integrated term by term (which is possible), ψ takes the form

$$\psi = a^2\lambda^{n-2}\left\{\frac{\zeta^2}{\rho^n}+\frac{1}{2(n-1)\rho^{n-2}}\right\}\sin n\theta - \frac{4nh^2\lambda^n \sin n\theta}{\lambda'^n}\left\{\int_0^\infty \frac{J_n(u\lambda')\,J_n(u\rho)}{u^3}\,du + 2\sum_{r=1}^{\infty}(-)^r \cos r\pi\zeta \int_0^\infty \frac{J_n(u\lambda')\,J_n(u\rho)}{u(u^2+r^2\pi^2)}\,du\right\}. \tag{4·6·2}$$

The integrals in (4·6·2) may be evaluated with the help of (4·2·31) and (4·4·4) and the result (Watson 1944, p. 401)

$$\int_0^\infty \frac{J_n(u\lambda)\,J_n(u\rho)}{u^3}\,du = \frac{\lambda^n}{8n(n-1)\rho^{n-2}} - \frac{\lambda^{n+2}}{8n(n+1)\rho^n} \quad (\rho \geq \lambda,\, n > 1). \tag{4·6·3}$$

Then, in addition, using the expansion

$$\zeta^2 = \tfrac{1}{3} + 4\sum_{r=1}^{\infty}\frac{(-)^r \cos r\pi\zeta}{r^2\pi^2} \quad (-1 \leq \zeta \leq 1), \tag{4·6·4}$$

(4·6·2) becomes

$$\psi = \frac{(2n+2+3\lambda'^2)\,h^2\lambda^n}{6(n+1)\rho^n}\sin n\theta + \frac{8nh^2\lambda^n \sin n\theta}{\pi^2\lambda'^n}\sum_{r=1}^{\infty}\frac{(-)^r I_n(r\pi\lambda')\,K_n(r\pi\rho)}{r^2}\cos r\pi\zeta. \tag{4·6·5}$$

The first term in (4·6·5) gives a stress system of the type called 'plane stress (a)' in § 4·1, while the remaining terms are a linear combination of potential functions of the type (4·3·2). This solution is therefore not independent and may be omitted. The solutions are then reduced to the ∞^3 solutions in § 4·3 to § 4·5, the solution given by (4·2·35) to (4·2·37), and the two plane stress solutions (a) and (b) given by (4·1·2) and (4·2·19). It can be shown that a linear relation exists between these ∞^3+3 solutions thus reducing the independent solutions to ∞^3+2 altogether as required. In order to obtain this linear relation consider the combination of potential functions

$$\frac{\psi_0+\omega_0+\phi_0}{8\eta n} + \sum_{m=1}^{\infty}(-)^m\left\{4(\omega_m+\phi_m)-(\omega'_m+\phi'_m+\chi_m)+\frac{n\psi_m}{m^2\pi^2\lambda^2}\right\}, \tag{4·6·6}$$

and the corresponding displacements when expressed as Fourier series in ζ, again restricting attention to the cases $n \geq 2$ for convenience. Then using the formula

$$8\sum_{m=1}^{\infty}\frac{u^3}{(u^2+m^2\pi^2)^2} = -\frac{4}{u}+\frac{\Sigma}{\sinh^2 u}, \tag{4·6·7}$$

which can be derived from (4·2·49) and (4·2·50), and assuming that orders of summations, and orders of integration and summations may be changed, it is found that the displacements may be written in the form

$$\frac{u_r}{h\cos n\theta} = \frac{\alpha+n-1}{4\eta n(n-1)}\frac{\lambda^n}{\rho^{n-1}} + 2\eta\int_0^\infty \frac{J_n(u\lambda)\,J_n'(u\rho)}{u^2}\,du$$
$$+2\sum_{r=1}^\infty\left\{\frac{n^2+\rho^2r^2\pi^2}{\rho r^2\pi^2}I_n(r\pi\lambda)\,K_n(r\pi\rho)+\lambda I_n'(r\pi\lambda)\,K_n'(r\pi\rho)-\frac{2(1-\eta)}{r\pi}I_n(r\pi\lambda)\,K_n'(r\pi\rho)\right.$$
$$\left.+2\int_0^\infty J_n(u\lambda)\,J_n'(u\rho)\left\{\frac{\eta}{u^2+r^2\pi^2}-\frac{r^2\pi^2}{(u^2+r^2\pi^2)^2}\right\}du\right\}(-)^r\cos r\pi\zeta, \quad (4\cdot6\cdot8)$$

$$\frac{u_\theta}{h\sin n\theta} = \frac{n-1-\alpha}{4\eta n(n-1)}\frac{\lambda^n}{\rho^{n-1}} - \frac{2\eta n}{\rho}\int_0^\infty \frac{J_n(u\lambda)\,J_n(u\rho)}{u^3}\,du$$
$$+2\sum_{r=1}^\infty\left\{\frac{2(1-\eta)\,n}{\rho r^2\pi^2}I_n(r\pi\lambda)\,K_n(r\pi\rho)-\frac{n}{\rho r\pi}[\lambda I_n'(r\pi\lambda)\,K_n(r\pi\rho)+\rho I_n(r\pi\lambda)\,K_n'(r\pi\rho)]\right.$$
$$\left.-\frac{2n}{\rho}\int_0^\infty \frac{J_n(u\lambda)\,J_n(u\rho)}{u}\left\{\frac{\eta}{u^2+r^2\pi^2}-\frac{r^2\pi^2}{(u^2+r^2\pi^2)^2}\right\}du\right\}(-)^r\cos r\pi\zeta, \quad (4\cdot6\cdot9)$$

$$\frac{u_z}{h\cos n\theta} = -2\sum_{r=1}^\infty\left\{\frac{2(1-\eta)}{r\pi}I_n(r\pi\lambda)\,K_n(r\pi\rho)+\lambda I_n'(r\pi\lambda)\,K_n(r\pi\rho)+\rho I_n(r\pi\lambda)\,K_n'(r\pi\rho)\right.$$
$$\left.+2r\pi\int_0^\infty \frac{J_n(u\lambda)\,J_n(u\rho)}{u}\left\{\frac{2-\eta}{u^2+r^2\pi^2}-\frac{r^2\pi^2}{(u^2+r^2\pi^2)^2}\right\}du\right\}(-)^r\sin r\pi\zeta. \quad (4\cdot6\cdot10)$$

With the help of (4·2·31), (4·4·4), (4·5·4), (4·6·4) and the formulae

$$\zeta = 2\sum_{r=1}^\infty \frac{(-)^{r+1}\sin r\pi\zeta}{r\pi} \quad (-1<\zeta<1), \quad (4\cdot6\cdot11)$$

$$\int_0^\infty \frac{J_n(u\lambda)\,J_n'(u\rho)}{u^2}\,du = -\frac{(n-2)\,\lambda^n}{8n(n-1)\,\rho^{n-1}}+\frac{\lambda^{n+2}}{8(n+1)\,\rho^{n+1}} \quad (\rho\geqslant\lambda, n>1), \quad (4\cdot6\cdot12)$$

$$m\pi\int_0^\infty \frac{J_n(u\lambda)\,J_n'(u\rho)}{u^2+m^2\pi^2}\,du = -\frac{\lambda^n}{2m\pi\rho^{n+1}}-I_n(m\pi\lambda)\,K_n'(m\pi\rho) \quad (\rho\geqslant\lambda), \quad (4\cdot6\cdot13)$$

$$2m^3\pi^3\int_0^\infty \frac{J_n(u\lambda)\,J_n'(u\rho)}{(u^2+m^2\pi^2)^2} = -\frac{\lambda^n}{m\pi\rho^{n+1}}-2I_n(m\pi\lambda)\,K_n'(m\pi\rho)$$
$$+\lambda m\pi I_n'(m\pi\lambda)\,K_n'(m\pi\rho)+\frac{n^2+\rho^2m^2\pi^2}{\rho m\pi}I_n(m\pi\lambda)\,K_n(m\pi\rho) \quad (\rho\geqslant\lambda), \quad (4\cdot6\cdot14)$$

the expressions (4·6·8) to (4·6·10) may be reduced to

$$\frac{u_r}{h\cos n\theta} = \frac{(1-\eta)\,\lambda^n}{8\eta n}\left\{\frac{4\eta n\zeta^2}{\rho^{n+1}}+\frac{7n+2-\alpha(n-2)}{2(n-1)\,\rho^{n-1}}\right\}+\left\{\frac{\eta}{4(n+1)}-\frac{1-\eta}{6\lambda^2}\right\}\frac{\lambda^{n+2}}{\rho^{n+1}}, \quad (4\cdot6\cdot15)$$

$$\frac{u_\theta}{h\sin n\theta} = \frac{(1-\eta)\,\lambda^n}{8\eta n}\left\{\frac{4\eta n\zeta^2}{\rho^{n+1}}+\frac{7n-16-\alpha n}{2(n-1)\,\rho^{n-1}}\right\}+\left\{\frac{\eta}{4(n+1)}-\frac{1-\eta}{6\lambda^2}\right\}\frac{\lambda^{n+2}}{\rho^{n+1}}, \quad (4\cdot6\cdot16)$$

$$\frac{u_z}{h\cos n\theta} = \frac{(1-\eta)\,\lambda^n\zeta}{n\rho^n}. \quad (4\cdot6\cdot17)$$

On referring to (4·1·4) and (4·2·20) it is seen that these displacements can be obtained from a linear combination of the plane stress solutions (*a*) and (*b*). Thus the linear relation between the ∞^3+3 solutions is given by

$$\frac{(\psi_0+\omega_0+\phi_0)}{8\eta n}-\frac{1-\eta}{8\eta n}(4\psi_b+\omega_b+\phi_b)+\left\{\frac{\eta}{4(n+1)}-\frac{1-\eta}{6\lambda^2}\right\}\frac{\omega_a}{n}$$
$$+\sum_{m=1}^{\infty}(-)^m\left\{4(\omega_m+\phi_m)-(\omega'_m+\phi'_m+\chi_m)+\frac{n\psi_m}{m^2\pi^2\lambda^2}\right\}=0. \quad (4·6·18)$$

5. Evaluation of the integrals

5·1. Before applying the above stress systems to a particular problem it is necessary to examine the integrals which appear in a rather involved way in the expressions for the displacements and stresses since it is desirable that their values should be known with some degree of accuracy. Attention will be confined to values of the displacements and stresses at the cylindrical hole, i.e. for $\rho=\lambda$, and for this value of ρ the integrals can all be expressed in the form

$$2\int_0^\infty \frac{u^{m-1}J_\mu(u\lambda)\,J_\nu(u\lambda)\sinh^2 u}{(u^2+k^2)^{n+1}\,\Sigma}\,du, \quad (5·1·1)$$

where m, μ, ν, n are all integers. These integrals could be expressed in series form by integration round a suitable contour but owing to the presence of the factor Σ in the denominator this is not a suitable form for numerical work. Instead, the factor $\Sigma^{-1}\sinh^2 u$ may be written in the form

$$\frac{\sinh^2 u}{\Sigma}=\frac{1-X}{2}, \quad (5·1·2)$$

where
$$X=(1+2u-e^{-2u})/\Sigma, \quad (5·1·3)$$

and the integral (5·1·1) may then be split up into two integrals

$$\int_0^\infty \frac{u^{m-1}J_\mu(u\lambda)\,J_\nu(u\lambda)}{(u^2+k^2)^{n+1}}\,du, \quad (5·1·4)$$

and
$$\int_0^\infty \frac{Xu^{m-1}J_\mu(u\lambda)\,J_\nu(u\lambda)}{(u^2+k^2)^{n+1}}\,du. \quad (5·1·5)$$

Consider first the integral (5·1·4). When $\nu-\mu<m<2n+3$ and $\mu+\nu+m$ is even the integral can be evaluated quite simply in terms of $I_\mu(\lambda k)$, $K_\nu(\lambda k)$ (Watson 1944, p. 429) but the cases required here are those for which $\mu+\nu+m$ is odd when no simple results appear to be available. The general integral (5·1·4) is, however, mentioned by Watson (1944, p. 436) and two methods are suggested for its evaluation. It appears to be of sufficient interest to outline both methods here.

5·2. In the first method the product $J_\mu(u\lambda)\,J_\nu(u\lambda)$ in the integrand of (5·1·4) is replaced by Neumann's integral (Watson 1944, p. 150)

$$J_\mu(u\lambda)\,J_\nu(u\lambda)=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi} J_{\mu+\nu}(2u\lambda\cos\theta)\cos(\mu-\nu)\,\theta\,d\theta. \quad (5·2·1)$$

Then, changing the order of integration (which can be justified under certain conditions) and evaluating the infinite integral by using a formula given by Watson (1944, p. 434), it is found that

$$\int_0^\infty \frac{u^{m-1} J_\mu(u\lambda)\, J_\nu(u\lambda)}{(u^2+k^2)^{n+1}}\,du$$
$$= \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} \cos(\mu-\nu)\,\theta \left\{\frac{(\lambda\cos\theta)^{\mu+\nu}\, k^{\mu+\nu+m-2n-2}\,\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}m)\,\Gamma(n+1-\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{2}m)}{\Gamma(n+1)\,\Gamma(\mu+\nu+1)}\right.$$
$$\times {}_1F_2\left(\frac{\mu+\nu+m}{2};\frac{\mu+\nu+m}{2}-n,\ \mu+\nu+1;\ k^2\lambda^2\cos^2\theta\right)$$
$$+\frac{(\lambda\cos\theta)^{2n+2-m}\,\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}m-n-1)}{\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+n+2-\frac{1}{2}m)}$$
$$\left.\times {}_1F_2\left(n+1;\frac{\mu+\nu-m}{2}+n+2,\ -\frac{\mu+\nu+m}{2}+n+2;\ k^2\lambda^2\cos^2\theta\right)\right\}d\theta, \quad (5{\cdot}2{\cdot}2)$$

where Γ is the Gamma function and ${}_1F_2$ is one of the generalized hypergeometric functions defined by

$$\left.\begin{aligned} {}_pF_q(\alpha_1,\alpha_2,\ldots,\alpha_p;\rho_1,\rho_2,\ldots,\rho_q;z) &= \sum_{r=0}^{\infty}\frac{(\alpha_1)_r\,(\alpha_2)_r\ldots(\alpha_p)_r}{r!\,(\rho_1)_r\,(\rho_2)_r\ldots(\rho_q)_r}z^r,\\ \alpha_r = \alpha(\alpha+1)\,(\alpha+2)\ldots(\alpha+r-1), &\quad (\alpha)_0 = 1. \end{aligned}\right\} \quad (5{\cdot}2{\cdot}3)$$

If term-by-term integration (which is possible in certain cases) is carried out in (5·2·2) by using the result

$$\int_0^{\frac{1}{2}\pi}\cos^{p+q-2}\theta\cos(p-q)\,\theta\,d\theta = \frac{\pi\Gamma(p+q-1)}{2^{p+q-1}\Gamma(p)\,\Gamma(q)} \quad (p+q>1), \quad (5{\cdot}2{\cdot}4)$$

the final value of the integral (5·1·4) can, after some reduction, be expressed in the form

$$\int_0^\infty \frac{u^{m-1} J_\mu(u\lambda)\, J_\nu(u\lambda)}{(u^2+k^2)^{n+1}}\,du$$
$$= \frac{\lambda^{\mu+\nu}k^{\mu+\nu+m-2n-2}\Gamma\left(\frac{\mu+\nu+m}{2}\right)\Gamma\left(n+1-\frac{\mu+\nu+m}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu}{2}+1\right)}{2\pi^{\frac{1}{2}}\Gamma(n+1)\,\Gamma(\mu+1)\,\Gamma(\nu+1)\,\Gamma(\mu+\nu+1)}$$
$$\times {}_3F_4\left(\frac{\mu+\nu+1}{2},\frac{\mu+\nu}{2}+1,\frac{\mu+\nu+m}{2};\mu+1,\nu+1,\mu+\nu+1,\frac{\mu+\nu+m}{2}-n;\lambda^2k^2\right)$$
$$+\frac{\lambda^{2n+2-m}\Gamma\left(\frac{3-m}{2}+n\right)\Gamma\left(n+2-\frac{m}{2}\right)\Gamma\left(\frac{\mu+\nu+m}{2}-n-1\right)}{2\pi^{\frac{1}{2}}\Gamma\left(n+2+\frac{\mu+\nu-m}{2}\right)\Gamma\left(n+2+\frac{\mu-\nu-m}{2}\right)\Gamma\left(n+2+\frac{\nu-\mu-m}{2}\right)}$$
$$\times {}_3F_4\left(n+1,\frac{3-m}{2}+n,n+2-\frac{m}{2};n+2+\frac{\mu+\nu-m}{2},n+2+\frac{\mu-\nu-m}{2},\right.$$
$$\left. n+2+\frac{\nu-\mu-m}{2},n+2-\frac{\mu+\nu+m}{2};\lambda^2k^2\right). \quad (5{\cdot}2{\cdot}5)$$

Although this result is true for certain complex values of λ, k, μ, ν, m, n attention is confined here to real values of these quantities and in this case they are restricted by the inequalities

$$\mu+\nu+m>0, \quad 2n+3>m. \quad (5{\cdot}2{\cdot}6)$$

The two series ${}_3F_4$ are convergent for all values of λk but they are only suitable for numerical computation if λk is not too large. Discussion of the evaluation of the integral for large values of λk is postponed until after the second method of obtaining the result (5·2·5) is shown in the next section.

Before proceeding to the next section a special case of (5·2·5) should be noted. When $n=-1$ the formula reduces considerably. The result for this case can also be found from Watson (1944, p. 403) and is

$$\int_0^\infty u^{m-1} J_\mu(u\lambda)\, J_\nu(u\lambda)\, du$$
$$= \frac{2^{m-1}\Gamma\left(\frac{\mu+\nu+m}{2}\right)\Gamma(1-m)}{\lambda^m\Gamma\left(1+\frac{\mu+\nu-m}{2}\right)\Gamma\left(1+\frac{\mu-\nu-m}{2}\right)\Gamma\left(1+\frac{\nu-\mu-m}{2}\right)} \quad \begin{pmatrix}\mu+\nu+m>0\\ 1>m\end{pmatrix}. \qquad (5\cdot2\cdot7)$$

5·3. In the second method the product $J_\mu(u\lambda)\, J_\nu(u\lambda)$ is replaced by the contour integral (Watson 1944, p. 436)

$$\frac{1}{2\pi i}\int_{-c-\infty i}^{-c+\infty i} \frac{\Gamma(-s)\,\Gamma(\mu+\nu+2s+1)\,(\frac{1}{2}\lambda u)^{\mu+\nu+2s}}{\Gamma(\mu+s+1)\,\Gamma(\nu+s+1)\,\Gamma(\mu+\nu+s+1)}\, ds, \qquad (5\cdot3\cdot1)$$

in which $\qquad \frac{\mu+\nu+m}{2}>c>0, \quad \frac{\mu+\nu+1}{2}>c>\frac{\mu+\nu+m}{2}-n-1,$

so that the poles of $\Gamma(-s)$ are on the right of the contour and those of $\Gamma(\mu+\nu+2s+1)$ are on the left. The order of integrations is then changed (which is possible for the cases considered here) and, making use of the result

$$\int_0^\infty \frac{u^{\mu+\nu+m+2s-1}}{(u^2+k^2)^{n+1}}\, du = \frac{\Gamma\left(\frac{\mu+\nu+m}{2}+s\right)\Gamma\left(n+1-\frac{\mu+\nu+m}{2}-s\right)k^{\mu+\nu+m+2s-2n-2}}{2\Gamma(n+1)}, \qquad (5\cdot3\cdot2)$$

the integral (5·1·4) takes the form

$$\int_0^\infty \frac{u^{m-1}J_\mu(u\lambda)\, J_\nu(u\lambda)}{(u^2+k^2)^{n+1}}\, du$$
$$= \frac{1}{4\pi i}\int_{-c-\infty i}^{-c+\infty i} \frac{\Gamma(-s)\,\Gamma(\mu+\nu+2s+1)\,\Gamma\left(\frac{\mu+\nu+m}{2}+s\right)\Gamma\left(n+1-\frac{\mu+\nu+m}{2}-s\right)(\frac{1}{2}\lambda)^{\mu+\nu+2s}\, k^{\mu+\nu+m+2s-2n-2}}{\Gamma(n+1)\,\Gamma(\mu+s+1)\,\Gamma(\nu+s+1)\,\Gamma(\mu+\nu+s+1)}\, ds. \qquad (5\cdot3\cdot3)$$

It may be shown from the asymptotic expansions of the Gamma function that the integrand of the contour integral in (5·3·3), when integrated round a semicircle of radius R with centre at $(-c, 0)$, on the right of the contour, tends to zero as $R\to\infty$ provided that R tends to infinity in such a manner that the semicircle never passes through any of the poles of the integrand. The integral may therefore be evaluated as the sum of the residues of

$$-\frac{\Gamma(-s)\,\Gamma(\mu+\nu+2s+1)\,\Gamma\left(\frac{\mu+\nu+m}{2}+s\right)\Gamma\left(n+1-\frac{\mu+\nu+m}{2}-s\right)(\frac{1}{2}\lambda)^{\mu+\nu+2s}\, k^{\mu+\nu+m+2s-2n-2}}{2\Gamma(n+1)\,\Gamma(\mu+s+1)\,\Gamma(\nu+s+1)\,\Gamma(\mu+\nu+s+1)} \qquad (5\cdot3\cdot4)$$

at the poles
$$s = r, \quad s = n+1-\frac{\mu+\nu+m}{2}+r, \tag{5·3·5}$$

where r takes all positive integral values including zero. Hence, remembering that the residue of $\Gamma(-z)$ at the pole $z = r$ is $(-)^{r+1}/r!$, the final result (5·2·5) is obtained once more, after some algebraic simplification.

5·4. As stated above, the result (5·2·5) is only convenient for numerical computation when λk is not too large, and since large as well as small values of λk occur in the applications it is necessary to see if an asymptotic expansion for the integral can be found. For this purpose a method due to Barnes (cf. Watson 1944, pp. 220, 351) can be used which employs a contour integral of the form (5·3·3). The cases which are required below are those for which $m = 1$ and $\mu+\nu$ is an even integer, and $m = 0$ with $\mu+\nu$ an odd integer, and it is convenient to consider these cases separately.

Take first $m = 1$, $\mu+\nu$ an even integer, μ and ν both being integers or zero, and consider the integral

$$\frac{1}{2\pi i}\int \frac{\Gamma(-s)\,\Gamma^2\left(\frac{\mu+\nu+1}{2}+s\right)\Gamma\left(\frac{\mu+\nu}{2}+1+s\right)\Gamma\left(\frac{1-\mu-\nu}{2}+n-s\right)\lambda^{\mu+\nu+2s}\,k^{\mu+\nu+2s-2n-1}}{2\pi^{\frac{1}{2}}\Gamma(n+1)\,\Gamma(\mu+1+s)\,\Gamma(\nu+1+s)\,\Gamma(\mu+\nu+1+s)}\,ds, \tag{5·4·1}$$

taken along the line $s = -\frac{1}{2}\mu-\frac{1}{2}\nu-1-p$ from $-\infty i$ to ∞i, where p is a large integer such that the poles of the integrand on the right of the contour are simple poles at the points

$$\left.\begin{array}{c} 0,\ 1,\ 2,\ \ldots \\ \frac{1-\mu-\nu}{2}+n,\quad \frac{1-\mu-\nu}{2}+n+1,\quad \frac{1-\mu-\nu}{2}+n+2,\ldots \end{array}\right\} \tag{5·4·2}$$

and double poles at the points

$$-\frac{\mu+\nu+1}{2},\quad -\frac{\mu+\nu+1}{2}-1,\ \ldots,\quad -\frac{\mu+\nu+1}{2}-p. \tag{5·4·3}$$

The integral (5·4·1) is $O(\lambda^{-2-2p}\,k^{-2n-3-2p})$ and, by a similar argument to that used in § 5·3, it follows that the integral is equal to the sum of the residues of

$$-\frac{\Gamma(-s)\,\Gamma^2\left(\frac{\mu+\nu+1}{2}+s\right)\Gamma\left(\frac{\mu+\nu}{2}+1+s\right)\Gamma\left(\frac{1-\mu-\nu}{2}+n-s\right)\lambda^{\mu+\nu+2s}\,k^{\mu+\nu+2s-2n-1}}{2\pi^{\frac{1}{2}}\Gamma(n+1)\,\Gamma(\mu+1+s)\,\Gamma(\nu+1+s)\,\Gamma(\mu+\nu+1+s)}, \tag{5·4·4}$$

at the poles (5·4·2) and (5·4·3). Thus, evaluating the residues with the help of the result that, when z is small and r is an integer or zero,

$$\Gamma(-r+z) = \frac{(-)^r}{r!\,z}\{1+z\psi(r)\}, \tag{5·4·5}$$

where
$$\psi(z) = \frac{d}{dz}\log\Gamma(z+1), \tag{5·4·6}$$

and remembering the formula (5·2·5), the following asymptotic expansion is found, after some algebraic reduction:

$$\int_0^\infty \frac{J_\mu(u\lambda)\,J_\nu(u\lambda)}{(u^2+k^2)^{n+1}}\,du \sim \frac{\cos\frac{1}{2}(\mu-\nu)\,\pi}{2\pi\lambda k^{2n+2}} \sum_{r=0}^{\infty} \frac{\left(\frac{\mu+\nu+1}{2}\right)_r \left(\frac{\mu-\nu+1}{2}\right)_r \left(\frac{\nu-\mu+1}{2}\right)_r \left(\frac{1-\mu-\nu}{2}\right)_r (n+1)_r}{(\frac{1}{2})_r\,(r!)^2\,(\lambda k)^{2r}}$$
$$\times\left\{\log(\lambda k)^2+2\psi(r)+\psi(r-\tfrac{1}{2})-\psi(r+n)-\psi\left(\frac{\mu+\nu-1}{2}+r\right)\right.$$
$$\left.-\psi\left(\frac{\mu-\nu-1}{2}+r\right)-\psi\left(\frac{\nu-\mu-1}{2}+r\right)-\psi\left(r-\frac{\mu+\nu+1}{2}\right)\right\}. \quad (5·4·7)$$

This expansion is valid for real values of μ, ν, λ, k, n such that μ and ν are integers and $\mu+\nu$ is an even positive integer, and $n+1>0$, although, if needed, it can be extended to certain complex values of λ, k, n.

5·5. In this section an asymptotic expansion of the integral (5·1·4) is found for large values of λk, for the case $m=0$ and $\mu+\nu$ equal to an odd integer, where μ and ν are both integers or zero, by considering the integral

$$\frac{1}{2\pi i}\int \frac{\Gamma(-s)\,\Gamma\left(\frac{\mu+\nu+1}{2}+s\right)\Gamma\left(\frac{\mu+\nu}{2}+1+s\right)\Gamma\left(\frac{\mu+\nu}{2}+s\right)\Gamma\left(n+1-\frac{\mu+\nu}{2}-s\right)\lambda^{\mu+\nu+2s}\,k^{\mu+\nu+2s-2n-2}}{2\pi^{\frac{1}{2}}\,\Gamma(n+1)\,\Gamma(\mu+1+s)\,\Gamma(\nu+1+s)\,\Gamma(\mu+\nu+1+s)}\,ds, \quad (5·5·1)$$

taken along the line $s=-\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{2}-p$ from $-\infty i$ to ∞i, where p is a large integer such that the poles of the integrand on the right of the contour are the simple poles

$$\left.\begin{array}{c} 0,\ 1,\ 2,\ \ldots \\ n+1-\frac{\mu+\nu}{2},\quad n+2-\frac{\mu+\nu}{2},\quad n+3-\frac{\mu+\nu}{2},\ \ldots;\quad -\frac{\mu+\nu}{2}, \end{array}\right\} \quad (5·5·2)$$

and double poles at the points

$$-\frac{\mu+\nu}{2}-1,\quad -\frac{\mu+\nu}{2}-2,\quad \ldots,\quad -\frac{\mu+\nu}{2}-p. \quad (5·5·3)$$

The remaining steps are now similar to those used in § 5·4, so that finally

$$\int_0^\infty \frac{J_\mu(u\lambda)\,J_\nu(u\lambda)}{u(u^2+k^2)^{n+1}}\,du \sim \frac{2\sin\frac{1}{2}(\mu-\nu)\,\pi}{\pi(\mu^2-\nu^2)\,k^{2n+2}} - \frac{(\mu^2-\nu^2)\,(n+1)}{4\pi\lambda^2k^{2n+4}\sin\frac{1}{2}(\mu-\nu)\,\pi}$$
$$\times\sum_{r=0}^{\infty}\frac{\left(\frac{\mu+\nu}{2}+1\right)_r\left(\frac{\mu-\nu}{2}+1\right)_r\left(\frac{\nu-\mu}{2}+1\right)_r\left(1-\frac{\mu+\nu}{2}\right)_r(n+2)_r}{(\frac{3}{2})_r\,r!\,(r+1)!\,(\lambda k)^{2r}}$$
$$\times\left\{\log(\lambda k)^2+\psi(r)+\psi(r+\tfrac{1}{2})+\psi(r+1)-\psi(r+n+1)-\psi\left(\frac{\mu+\nu}{2}+r\right)\right.$$
$$\left.-\psi\left(\frac{\mu-\nu}{2}+r\right)-\psi\left(\frac{\nu-\mu}{2}+r\right)-\psi\left(r-\frac{\mu+\nu}{2}\right)\right\}, \quad (5·5·4)$$

provided μ and ν are integers, $\mu+\nu$ is an odd positive integer and $2n+3>0$.

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5·6. In the tension problem which is discussed below the integrals of the type (5·1·4) which are required can be reduced to eight integrals

$$L_{n+1}(\lambda,k)=\int_0^\infty \frac{J_2^2(u\lambda)}{(u^2+k^2)^{n+1}}\,du,\quad M_{n+1}(\lambda,k)=\int_0^\infty \frac{J_2(u\lambda)\,J_1(u\lambda)}{u(u^2+k^2)^{n+1}}\,du\quad (n=0,1,2,3),\qquad (5·6·1)$$

where k has the values $r\pi$, r being a positive integer. These integrals can be expressed as convergent power series in $\lambda r\pi$ by using the formula (5·2·5), and they reduce to the following expressions:

$$L_1(\lambda,r\pi)=\frac{4\lambda}{15\pi}\,{}_2F_3(1,1;\,-\tfrac{1}{2},1\tfrac{1}{2},3\tfrac{1}{2};\,r^2\pi^2\lambda^2)+\frac{r^3\pi^4\lambda^4}{128}\,{}_1F_2(2\tfrac{1}{2};\,3,5;\,r^2\pi^2\lambda^2),\qquad (5·6·2)$$

$$L_2(\lambda,r\pi)=\frac{32\lambda^3}{315\pi}\,{}_2F_3(2,2;\,\tfrac{1}{2},2\tfrac{1}{2},4\tfrac{1}{2};\,r^2\pi^2\lambda^2)-\frac{3r\pi^2\lambda^4}{256}\,{}_2F_3(2\tfrac{1}{2},2\tfrac{1}{2};\,1\tfrac{1}{2},3,5;\,r^2\pi^2\lambda^2),\qquad (5·6·3)$$

$$L_3(\lambda,r\pi)=-\frac{512\lambda^5}{14175\pi}\,{}_2F_3(3,3;\,1\tfrac{1}{2},3\tfrac{1}{2},5\tfrac{1}{2};\,r^2\pi^2\lambda^2)+\frac{3\lambda^4}{1024r}\,{}_2F_3(2\tfrac{1}{2},2\tfrac{1}{2};\,\tfrac{1}{2},3,5;\,r^2\pi^2\lambda^2),\qquad (5·6·4)$$

$$L_4(\lambda,r\pi)=\frac{2^{12}\lambda^7}{99(105)^2\pi}\,{}_2F_3(4,4;\,2\tfrac{1}{2},4\tfrac{1}{2},6\tfrac{1}{2};\,r^2\pi^2\lambda^2)+\frac{\lambda^4}{2048\pi^2r^3}\,{}_2F_3(2\tfrac{1}{2},2\tfrac{1}{2};\,-\tfrac{1}{2},3,5;\,r^2\pi^2\lambda^2),\qquad (5·6·5)$$

$$M_1(\lambda,r\pi)=-\frac{r\pi^2\lambda^3}{32}\,{}_1F_2(2\tfrac{1}{2};\,3,4;\,r^2\pi^2\lambda^2)+\frac{16\lambda^2}{45\pi}\,{}_2F_3(1,2;\,\tfrac{1}{2},2\tfrac{1}{2},3\tfrac{1}{2};\,r^2\pi^2\lambda^2),\qquad (5·6·6)$$

$$M_2(\lambda,r\pi)=\frac{\lambda^3}{64r}\,{}_2F_3(1\tfrac{1}{2},2\tfrac{1}{2};\,\tfrac{1}{2},3,4;\,r^2\pi^2\lambda^2)-\frac{256\lambda^4}{1575\pi}\,{}_2F_3(2,3;\,1\tfrac{1}{2},3\tfrac{1}{2},4\tfrac{1}{2};\,r^2\pi^2\lambda^2),\qquad (5·6·7)$$

$$M_3(\lambda,r\pi)=\frac{\lambda^3}{256\pi^2r^3}\,{}_2F_3(1\tfrac{1}{2},2\tfrac{1}{2};\,-\tfrac{1}{2},3,4;\,r^2\pi^2\lambda^2)+\frac{2^{11}\lambda^6}{9(105)^2\pi}\,{}_2F_3(3,4;\,2\tfrac{1}{2},4\tfrac{1}{2},5\tfrac{1}{2};\,r^2\pi^2\lambda^2),\qquad (5·6·8)$$

$$M_4(\lambda,r\pi)=\frac{\lambda^3}{512\pi^4r^5}\,{}_2F_3(1\tfrac{1}{2},2\tfrac{1}{2};\,-1\tfrac{1}{2},3,4;\,r^2\pi^2\lambda^2)-\frac{2^{16}\lambda^8}{55(945)^2\pi}\,{}_2F_3(4,5;\,3\tfrac{1}{2},5\tfrac{1}{2},6\tfrac{1}{2};\,r^2\pi^2\lambda^2).\qquad (5·6·9)$$

Numerical work is restricted to the values $\lambda=1$ and $r=1,2,3,4,5,6$. When $\lambda=1$ and $r=1$, the series in (5·6·2) to (5·6·9) have been evaluated to ten significant figures, about fifteen terms of each series being required for this accuracy. The integrals can then only be found to seven significant figures (with the seventh figure doubtful) since each integral consists of the difference of two terms whose first three figures are usually the same. Values of the integrals are recorded in tables 1 and 2.

When $\lambda=1$ and $r=2,3,4,5,6$, the above method of computation is impossibly laborious and also not very accurate since the integrals are expressed as the difference of two nearly equal quantities. It is, however, possible to make use of the asymptotic expansions (5·4·7) and (5·5·4), which for the integrals (5.6.1) become

$$L_{n+1}(\lambda,r\pi)\sim\frac{1}{2\pi\lambda(r\pi)^{2n+2}}\sum_{m=0}^{\infty}\frac{(-1\tfrac{1}{2})_m(\tfrac{1}{2})_m(2\tfrac{1}{2})_m(n+1)_m}{(m!)^2(\lambda r\pi)^{2m}}\{\log(r\pi)^2+2\psi(m)-\psi(m-2\tfrac{1}{2})-\psi(m-\tfrac{1}{2})-\psi(m+1\tfrac{1}{2})-\psi(m+n)\}\quad(n\geqslant 0),\qquad (5·6·10)$$

$$M_{n+1}(\lambda,r\pi)\sim\frac{2}{3\pi(r\pi)^{2n+2}}-\frac{3(n+1)}{4\pi\lambda^2(r\pi)^{2n+4}}\sum_{m=0}^{\infty}\frac{(-\tfrac{1}{2})_m(\tfrac{1}{2})_m(2\tfrac{1}{2})_m(n+2)_m}{m!(m+1)!(\lambda r\pi)^{2m}}\{\log(r\pi)^2+\psi(m)+\psi(m+1)-\psi(m-1\tfrac{1}{2})-\psi(m-\tfrac{1}{2})-\psi(m+1\tfrac{1}{2})-\psi(m+n+1)\}\quad(n\geqslant 0).\qquad (5·6·11)$$

When using these results the following values of the ψ function are needed, namely:

$$\left.\begin{aligned}\psi(0) &= -\gamma = -0{\cdot}5772156649,\\ \psi(-\tfrac{1}{2}) &= -\gamma-2\log 2,\\ \psi(z+1) &= \psi(z)+\frac{1}{z+1},\\ \psi(-z) &= \psi(z-1)+\pi\cot\pi z.\end{aligned}\right\} \quad (5{\cdot}6{\cdot}12)$$

The asymptotic formulae (5·6·10) and (5·6·11) have been used to evaluate the integrals (5·6·1) for the cases $\lambda = 1$, $r = 2, 3, 4, 5, 6$. It is not possible to give a precise estimate of the accuracy of these formulae. For the larger values of r it is easy to obtain accuracy to seven significant figures. The accuracy is much less certain for $r = 3$ and $n = 2$ and 3, and for $r = 2$ and all the relevant values of n. Numerical results are given in tables 1 and 2. Seven figures are given for $r \geqslant 3$ but the seventh figure, and sometimes the sixth may be unreliable, particularly for $n = 2$ and 3. When $r = 2$ the fifth figure, and possibly the fourth when $n = 2$ and 3, may be unreliable.

Table 1. Values of $L_{n+1}(1, r\pi)$

n \ r	1	2	3	4	5	6
0	$3{\cdot}300107\times10^{-2}$	$1{\cdot}42075\times10^{-2}$	$7{\cdot}864947\times10^{-3}$	$5{\cdot}030688\times10^{-3}$	$3{\cdot}516991\times10^{-3}$	$2{\cdot}609720\times10^{-3}$
1	$1{\cdot}691895\times10^{-3}$	$2{\cdot}4977\times10^{-4}$	$6{\cdot}726895\times10^{-5}$	$2{\cdot}523094\times10^{-5}$	$1{\cdot}156626\times10^{-5}$	$6{\cdot}057183\times10^{-6}$
2	$1{\cdot}010047\times10^{-4}$	$4{\cdot}9054\times10^{-6}$	$6{\cdot}340276\times10^{-7}$	$1{\cdot}382816\times10^{-7}$	$4{\cdot}132380\times10^{-8}$	$1{\cdot}520712\times10^{-8}$
3	$6{\cdot}49220\times10^{-6}$	$1{\cdot}0055\times10^{-7}$	$6{\cdot}19692\times10^{-9}$	$7{\cdot}833330\times10^{-10}$	$1{\cdot}522491\times10^{-10}$	$3{\cdot}930249\times10^{-11}$

Table 2. Values of $M_{n+1}(1, r\pi)$

n \ r	1	2	3	4	5	6
0	$1{\cdot}483365\times10^{-2}$	$4{\cdot}7253\times10^{-3}$	$2{\cdot}234914\times10^{-3}$	$1{\cdot}289402\times10^{-3}$	$8{\cdot}359726\times10^{-4}$	$5{\cdot}849453\times10^{-4}$
1	$1{\cdot}104986\times10^{-3}$	$1{\cdot}0736\times10^{-4}$	$2{\cdot}377794\times10^{-5}$	$7{\cdot}882577\times10^{-6}$	$3{\cdot}306650\times10^{-6}$	$1{\cdot}617062\times10^{-6}$
2	$8{\cdot}538238\times10^{-5}$	$2{\cdot}4619\times10^{-6}$	$2{\cdot}541604\times10^{-7}$	$4{\cdot}832645\times10^{-8}$	$1{\cdot}310444\times10^{-8}$	$4{\cdot}476492\times10^{-9}$
3	$6{\cdot}793096\times10^{-6}$	$5{\cdot}6794\times10^{-8}$	$2{\cdot}724623\times10^{-9}$	$2{\cdot}968129\times10^{-10}$	$5{\cdot}199728\times10^{-11}$	$1{\cdot}240321\times10^{-11}$

The special cases of (5·2·7) which are required are

$$\left.\begin{aligned}&\int_0^\infty \frac{J_2^2(u\lambda)}{u^2}\,du = \frac{4\lambda}{15\pi}, \qquad \int_0^\infty \frac{J_2^2(u\lambda)}{u^4}\,du = \frac{32\lambda^3}{315\pi},\\ &\int_0^\infty \frac{J_2(u\lambda)\,J_1(u\lambda)}{u} = \frac{2}{3\pi}, \quad \int_0^\infty \frac{J_2(u\lambda)\,J_1(u\lambda)}{u^3}\,du = \frac{16\lambda^2}{45\pi}.\end{aligned}\right\} \quad (5{\cdot}6{\cdot}13)$$

5·7. A partial check on the numerical results in tables 1 and 2 may be obtained by using some relations which are found to exist among integrals of the types (5·6·1). Consider the slightly more general integrals

$$\left.\begin{aligned}&L_n(\mu,\lambda,k) = \int_0^\infty \frac{J_\mu^2(u\lambda)}{(u^2+k^2)^n}\,du, \quad N_n(\mu,\lambda,k) = \int_0^\infty \frac{J_{\mu-1}^2(u\lambda)}{(u^2+k^2)^n}\,du,\\ &M_n(\mu,\lambda,k) = \int_0^\infty \frac{J_\mu(u\lambda)\,J_{\mu-1}(u\lambda)}{u(u^2+k^2)^n}\,du,\end{aligned}\right\} \quad (5{\cdot}7{\cdot}1)$$

where $\mu \geqslant 1$. By integration by parts and by using the relations

$$\left.\begin{aligned}\lambda u\,J_\mu'(u\lambda) &= \lambda u\,J_{\mu-1}(u\lambda) - \mu\,J_\mu(u\lambda),\\ \lambda u\,J_{\mu-1}'(u\lambda) &= (\mu-1)\,J_{\mu-1}(u\lambda) - \lambda u\,J_\mu(u\lambda),\end{aligned}\right\} \quad (5{\cdot}7{\cdot}2)$$

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the following formulae may be obtained:

$$\left.\begin{aligned} 2nk^2L_{n+1} &= (2n+2\mu-1)\,L_n - 2\lambda(M_{n-1}-k^2M_n),\\ 2nk^2N_{n+1} &= (2n-2\mu+1)\,N_n + 2\lambda(M_{n-1}-k^2M_n),\\ \lambda N_n &= \lambda L_n + M_n + 2n(M_n - k^2M_{n+1}). \end{aligned}\right\} \quad (5\cdot7\cdot3)$$

The elimination of N then gives

$$\left.\begin{aligned} 2nk^2L_{n+1} &= (2n+2\mu-1)\,L_n + 2\lambda(k^2M_n - M_{n-1}),\\ 4n(n+1)\,k^4M_{n+2} &= 4n(2n+2-\mu)\,k^2M_{n+1} - (2n+1)\,(2n-2\mu+1)\,M_n\\ &\quad + 4nk^2\lambda L_{n+1} - 4n\lambda L_n, \end{aligned}\right\} \quad (5\cdot7\cdot4)$$

for $\mu \geqslant 1$, $n \geqslant 1$. Relations between the integrals (5·6·1) are obtained by putting $\mu = 2$. From these formulae the values of L_2, L_3, L_4, M_3, M_4 can be found in terms of L_1, M_0 and M_1. This has been done for the integrals whose values are recorded in tables 1 and 2 and good agreement was obtained with the values given in the table which were found by direct calculation.

5·8. Integrals of the type (5·1·5) which are required can be reduced to the eight integrals

$$S_{n+1}(\lambda, k) = \int_0^\infty \frac{XJ_2^2(u\lambda)}{(u^2+k^2)^{n+1}}\,du, \quad T_{n+1}(\lambda, k) = \int_0^\infty \frac{XJ_2(u\lambda)\,J_1(u\lambda)}{u(u^2+k^2)^{n+1}}\,du \quad (n = 0, 1, 2, 3), \quad (5\cdot8\cdot1)$$

where k has the values $r\pi$, r being a positive integer. No explicit expression has been found for these integrals owing to the presence of the factor X in the integrand. The integrand, however, contains negative exponential terms when the variable u is large so that evaluation of the integrals can be effected by numerical integration. This has been done for the case $\lambda = 1$ and the results are recorded in tables 3 and 4.

Table 3. Values of $S_{n+1}(1, r\pi)$

$n \backslash r$	1	2	3	4	5	6
0	$2\cdot4967\times10^{-3}$	$7\cdot9998\times10^{-4}$	$3\cdot7712\times10^{-4}$	$2\cdot1689\times10^{-4}$	$1\cdot4028\times10^{-4}$	$9\cdot7985\times10^{-5}$
1	$1\cdot8294\times10^{-4}$	$1\cdot8217\times10^{-5}$	$4\cdot0350\times10^{-6}$	$1\cdot3336\times10^{-6}$	$5\cdot5776\times10^{-7}$	$2\cdot7211\times10^{-7}$
2	$1\cdot3816\times10^{-5}$	$4\cdot1656\times10^{-7}$	$4\cdot3216\times10^{-8}$	$8\cdot2026\times10^{-9}$	$2\cdot2180\times10^{-9}$	$7\cdot5572\times10^{-10}$
3	$1\cdot0710\times10^{-6}$	$9\cdot5622\times10^{-9}$	$4\cdot6331\times10^{-10}$	$5\cdot0470\times10^{-11}$	$8\cdot8215\times10^{-12}$	$2\cdot0990\times10^{-12}$

Table 4. Values of $T_{n+1}(1, r\pi)$

$n \backslash r$	1	2	3	4	5	6
0	$3\cdot6622\times10^{-3}$	$1\cdot0530\times10^{-3}$	$4\cdot8274\times10^{-4}$	$2\cdot7465\times10^{-4}$	$1\cdot7672\times10^{-4}$	$1\cdot2308\times10^{-4}$
1	$3\cdot1045\times10^{-4}$	$2\cdot5235\times10^{-5}$	$5\cdot2952\times10^{-6}$	$1\cdot7135\times10^{-6}$	$7\cdot0934\times10^{-7}$	$3\cdot4409\times10^{-7}$
2	$2\cdot6751\times10^{-5}$	$6\cdot0584\times10^{-7}$	$5\cdot8109\times10^{-8}$	$1\cdot0691\times10^{-8}$	$2\cdot8475\times10^{-9}$	$9\cdot6196\times10^{-10}$
3	$2\cdot3378\times10^{-6}$	$1\cdot4571\times10^{-8}$	$6\cdot3795\times10^{-10}$	$6\cdot6721\times10^{-11}$	$1\cdot1431\times10^{-11}$	$2\cdot6894\times10^{-12}$

For the S integrals the integrands were tabulated for values of u from 0 to 7·2 at intervals of 0·2. The values of $J_2(u)$ were found to ten figures from $J_1(u)$ and $J_0(u)$ which are given by Gray, Mathews & MacRobert (1922, p. 267). The values of X were found to ten figures for $u = 0$ to 3·6 by using Newman's tables of the exponential function (1883). For $u = 3\cdot8$ to 7·2 X was evaluated to seven figures. The S integrals for the cases $\lambda = 1$, and $r = 1, 2, 3, 4$ were then all evaluated by applying Weddle's Rule (Whittaker & Robinson 1946, p. 151) which is correct to fifth differences, to each sixth of the range. The results were all checked by applying the Newton-Cotes formula (Whittaker & Robinson 1946, p. 152) to each seventh of the range $u = 0$ to 7. There was always agreement in the first five figures. The contributions to the integrals from the range $u = 7{\cdot}2$ to infinity do not affect the first five figures and were therefore neglected.

The same procedure was adopted for the T integrals but for five figure accuracy, or at most an error of 1 in the fifth figure, it was only necessary to tabulate the integrand for values of u from 0 to 6.

For $\lambda = 1$ and for large values of r asymptotic formulae for the integrals can be obtained which reduce the labour of computation. The required formulae are found by expanding the factor $(u^2+k^2)^{-(n+1)}$ in the integrands in inverse powers of k^2 and integrating term by term. It is then necessary to know the values of integrals of the form

$$ {}^mS_0(\lambda) = \int_0^\infty Xu^{2m} J_2^2(u\lambda)\,du, \quad {}^mT_0(\lambda) = \int_0^\infty Xu^{2m-1} J_2(u\lambda)\,J_1(u\lambda)\,du, \tag{5·8·2} $$

for $m = 0, 1, 2, 3, \ldots$. The integrals ${}^mS_0(1)$, ${}^mT_0(1)$ for $m = 0, 1, 2$ and 3 have been evaluated by numerical integration using a range of values of u from 0 to 8 at intervals of 0·2, the contributions from the rest of the range being neglected. Thus

$$ \left. \begin{aligned} {}^0S_0(1) &= 0{\cdot}035287, & {}^1S_0(1) &= 0{\cdot}17093, \\ {}^2S_0(1) &= 1{\cdot}191, & {}^3S_0(1) &= 11{\cdot}09, \\ {}^0T_0(1) &= 0{\cdot}044028, & {}^1T_0(1) &= 0{\cdot}10650, \\ {}^2T_0(1) &= 0{\cdot}4262, & {}^3T_0(1) &= 2{\cdot}296. \end{aligned} \right\} \tag{5·8·3} $$

Using these values it is found that the first four terms in the asymptotic expansions of the integrals (5·8·1) for large values of k are

$$ \int_0^\infty \frac{XJ_2^2(u)}{(u^2+k^2)^n}\,du \sim \frac{0{\cdot}035287}{k^2}\left\{1 - \frac{4{\cdot}8442n}{k^2} + \frac{16{\cdot}88n(n+1)}{k^4} - \frac{52{\cdot}40n(n+1)\,(n+2)}{k^6} + \ldots\right\} \tag{5·8·4} $$

and

$$ \int_0^\infty \frac{XJ_2(u)\,J_1(u)}{u(u^2+k^2)^n}\,du \sim \frac{0{\cdot}044028}{k^2}\left\{1 - \frac{2{\cdot}4188n}{k^2} + \frac{4{\cdot}8403n(n+1)}{k^4} - \frac{8{\cdot}690n(n+1)\,(n+2)}{k^6} + \ldots\right\}. \tag{5·8·5} $$

As a check on these asymptotic formulae the integrals (5·8·1) were re-evaluated for the case $\lambda = 1$, $k = 4\pi$. The results agreed, or at most differed by 1 in the fifth figure, with those found previously, except for the integrals $S_3(1, 4\pi)$ and $S_4(1, 4\pi)$; but even in these integrals the results only differed by about 7 and 10 respectively in the fifth figure. It appears, therefore, that the asymptotic formulae (5·8·4) and (5·8·5) may safely be used for the values $k = 5\pi$ and 6π (and greater values). Results found in this way are included in tables 3 and 4.

6. The tension problem

6·1. Attention is now directed to a particular problem in which a plate containing a circular cylindrical hole, free from applied stress, is acted on by a uniform tension T at large distances from the hole. If the hole is absent the only non-zero component of stress is $\widehat{xx}$

which is equal to T, corresponding to a uniform tension parallel to the x-axis. In cylindrical polar co-ordinates the stresses are

$$\widehat{rr} = \tfrac{1}{2}T(1+\cos 2\theta), \quad \widehat{r\theta} = -\tfrac{1}{2}T\sin 2\theta, \quad \widehat{\theta\theta} = \tfrac{1}{2}T(1-\cos 2\theta). \tag{6·1·1}$$

These stresses satisfy the boundary conditions (2·1·1) on the faces of the plate but they do not give zero values for the normal and shear stresses when $\rho = \lambda$. The constant part of the normal stress $\widehat{rr}$, namely $\frac{1}{2}T$, may be removed at the hole $\rho = \lambda$ by adding a stress function ω of type B, in the form

$$\omega = \frac{Ta^2}{4\mu}\log \rho, \tag{6·1·2}$$

with corresponding stresses $$\widehat{rr} = -\frac{T\lambda^2}{2\rho^2}, \quad \widehat{\theta\theta} = \frac{T\lambda^2}{2\rho^2}, \tag{6·1·3}$$

the remaining components of stress being zero. Thus, the parts of the stresses which are independent of the angular co-ordinate θ are

$$\widehat{rr} = \tfrac{1}{2}T(1-\lambda^2/\rho^2), \quad \widehat{\theta\theta} = \tfrac{1}{2}T(1+\lambda^2/\rho^2), \tag{6·1·4}$$

the remaining components of stress being zero. In particular, at the hole $\rho = \lambda$, $\widehat{\theta\theta} = T$.

To complete the solution it is now necessary to find a system of stresses which have zero values at infinity and zero values for the normal and shear stresses on the faces of the plate, and which, with the stresses

$$\widehat{rr} = \tfrac{1}{2}T\cos 2\theta, \quad \widehat{r\theta} = -\tfrac{1}{2}T\sin 2\theta, \quad \widehat{\theta\theta} = -\tfrac{1}{2}T\cos 2\theta, \tag{6·1·5}$$

have zero values for the normal and shear stresses at the hole. To do this the stresses which are derived from the following potential functions are added to those given in (6·1·5), namely:

$$\frac{A_0\omega_a}{12\mu} + \frac{B_0}{2\mu}\{k(\psi_0+\omega_0+\phi_0) + k'(4\psi_b+\omega_b+\phi_b)\}$$
$$+\frac{1}{2\mu}\sum_{m=1}^{\infty}\{A_m\psi_m + B_m(\omega_m+\phi_m) + D_m\{4(\omega_m+\phi_m)-(\omega'_m+\phi'_m+\chi_m)\}\}, \tag{6·1·6}$$

where throughout n takes the value 2 and where A_m, B_m, D_m, k, k' are constants which are to be determined from the boundary conditions at the hole, the conditions at the faces of the plate being already satisfied by the stresses which correspond to (6·1·6). One more constant than is necessary for the satisfaction of the boundary conditions at the hole is included in (6·1·6) for a reason which will appear later. It is found to be more convenient to take $4(\omega_m+\phi_m)-(\omega'_m+\phi'_m+\chi_m)$ instead of $\omega'_m+\phi'_m+\chi_m$ as a set of fundamental potential functions.

If the values of $\widehat{rr}$, $\widehat{r\theta}$ and $\widehat{rz}$ which correspond to (6·1·6) are obtained in the form of Fourier series in ζ, cosine series for $\widehat{rr}$ and $\widehat{r\theta}$ and a sine series for $\widehat{rz}$, then the condition that these stresses, when added to those given in (6·1·5), should be zero when $\rho = \lambda$ can be satisfied by equating to zero each coefficient of $\cos r\pi\zeta$ and $\sin r\pi\zeta$. The required Fourier series are found from (4·1·4), (4·2·21), (4·2·54), (4·3·3), (4·4·19) and (4·5·20), where in (4·2·21)

ζ^2 is expanded with the help of (4·6·4). Thus, assuming that the order of summations may be changed, the following set of equations is found for the constants:

$$\left.\begin{aligned} &A_0+(\tfrac{1}{2}\,{}^0a_0-8)\,kB_0-8(1+\eta+\eta\lambda^{-2})\,k'B_0+\tfrac{1}{2}T \\ &\qquad\qquad+\frac{1}{2}\sum_{m=1}^{\infty}\{{}^ma_0\,B_m+(4\,{}^ma_0-{}^mu_0)\,D_m\}=0, \\ &A_0+(\tfrac{1}{2}\,{}^0b_0-4)\,kB_0-4(1+\eta+2\eta\lambda^{-2})\,k'B_0-\tfrac{1}{2}T \\ &\qquad\qquad+\frac{1}{2}\sum_{m=1}^{\infty}\{{}^mb_0\,B_m+(4\,{}^mb_0-{}^mv_0)\,D_m\}=0, \end{aligned}\right\} \tag{6·1·7}$$

and

$$\left.\begin{aligned} &\left\{{}^0a_r\,k-\frac{96\eta(-)^r k'}{r^2\pi^2\lambda^2}\right\}B_0+g_r\,A_r+a'_rB_r+(4a'_r-u'_r)\,D_r \\ &\qquad\qquad+\sum_{m=1}^{\infty}\{{}^ma_r\,B_m+(4\,{}^ma_r-{}^mu_r)\,D_m\}=0, \\ &\left\{{}^0b_r\,k-\frac{96\eta(-)^r k'}{r^2\pi^2\lambda^2}\right\}B_0+h_r\,A_r+b'_rB_r+(4b'_r-v'_r)\,D_r \\ &\qquad\qquad+\sum_{m=1}^{\infty}\{{}^mb_r\,B_m+(4\,{}^mb_r-{}^mv_r)\,D_m\}=0, \\ &{}^0c_r\,kB_0+i_r\,A_r+c'_rB_r+(4c'_r-w'_r)\,D_r \\ &\qquad\qquad+\sum_{m=1}^{\infty}\{{}^mc_r\,B_m+(4\,{}^mc_r-{}^mw_r)\,D_m\}=0, \end{aligned}\right\} \tag{6·1·8}$$

for all integral values of r. The constants A_r may easily be eliminated from (6·1·8) so that the equations for B_r and D_r become

$$\left.\begin{aligned} &\left\{\left({}^0a_r\,k-\frac{96\eta(-)^r k'}{r^2\pi^2\lambda^2}\right)h_r-\left({}^0b_r\,k-\frac{96\eta(-)^r k'}{r^2\pi^2\lambda^2}\right)g_r\right\}B_0 \\ &\quad+(a'_rh_r-b'_rg_r)\,B_r+\{(4a'_r-u'_r)\,h_r-(4b'_r-v'_r)\,g_r\}\,D_r \\ &\quad+\sum_{r=1}^{\infty}\{({}^ma_rh_r-{}^mb_rg_r)\,B_m+\{(4\,{}^ma_r-{}^mu_r)\,h_r-(4\,{}^mb_r-{}^mv_r)\,g_r\}\,D_m\}=0, \\ &\left\{\left({}^0b_r\,k-\frac{96\eta(-)^r k'}{r^2\pi^2\lambda^2}\right)i_r-{}^0c_r\,kh_r\right\}B_0 \\ &\quad+(b'_ri_r-c'_rh_r)\,B_r+\{(4b'_r-v'_r)\,i_r-(4c'_r-w'_r)\,h_r\}\,D_r \\ &\quad+\sum_{r=1}^{\infty}\{({}^mb_ri_r-{}^mc_rh_r)\,B_m+\{(4\,{}^mb_r-{}^mv_r)\,i_r-(4\,{}^mc_r-{}^mw_r)\,h_r\}\,D_m\}=0. \end{aligned}\right\} \tag{6·1·9}$$

6·2. Numerical calculations are inevitably extremely heavy owing to the complicated nature of the functions which are used, so attention has had to be confined to an example which is likely to be one of particular interest. When the plate is infinitely thick ($\lambda\to 0$) the problem becomes one of plane strain and the solution is well known. Again, when the plate is very thin ($\lambda\to\infty$) the problem is one of plane stress with a solution similar in form to that of plane strain. In both of these cases the maximum stress concentration at the hole is $3T$ at the ends of a diameter perpendicular to the applied tension. The generalized plane stress theory gives a maximum value of $3T$ for the average stress concentration at the hole and this result is usually supposed to be valid for moderately thick plates. It would appear therefore that an example of some interest is that which lies about mid-way between the extreme cases $\lambda \to 0$, $\lambda \to \infty$, so calculations have been carried out for $\lambda = 1$, i.e. when the diameter of the hole is equal to the thickness of the plate. The value $\eta = 0{\cdot}25$ has been chosen for Poisson's ratio.

Some of the constants in (6·1·7) and (6·1·9) depend on the integrals which were evaluated in § 5·6 to § 5·8 and whose values are given in tables 1 to 4. The remaining constants depend on the Bessel functions $I_2(r\pi\lambda)$, $K_2(r\pi\lambda)$ which can be expressed in terms of $I_0(r\pi\lambda)$, $I_1(r\pi\lambda)$; $K_0(r\pi\lambda)$, $K_1(r\pi\lambda)$. Values of $e^{-x}I_0(x)$, $e^{-x}I_1(x)$, $e^{x}K_0(x)$, $e^{x}K_1(x)$ are recorded in Watson (1944, p. 698) for $x = 0$ to 16 at intervals of 0·2 and these are suitable for interpolation and have been used to obtain results for $r = 1, 2, 3, 4, 5$, $\lambda = 1$. Since only products of the I and K functions are required the exponential factor need not be evaluated. When $r = 6$ values of $I_2(6\pi)$ and $K_2(6\pi)$ were found from the asymptotic expansions for these functions. For reference the values of $I_0(r\pi\lambda)$, $I_1(r\pi\lambda)$, $I_2(r\pi\lambda)$, $K_0(r\pi\lambda)$, $K_1(r\pi\lambda)$, $K_2(r\pi\lambda)$ for $\lambda = 1$, $r = 1, 2, 3, 4, 5, 6$ are given in table 5.

TABLE 5

r	$e^{-r\pi}I_0(r\pi)$	$e^{-r\pi}I_1(r\pi)$	$e^{-r\pi}I_2(r\pi)$	$e^{r\pi}K_0(r\pi)$	$e^{r\pi}K_1(r\pi)$	$e^{r\pi}K_2(r\pi)$
1	0·2367192	0·1940935	0·1131554	0·6828514	0·7847820	1·182459
2	0·1626701	0·1490947	0·1152118	0·4908233	0·5285282	0·6590591
3	0·1317896	0·1245885	0·1053511	0·4031260	0·4239971	0·4931010
4	0·1137138	0·1090910	0·09635143	0·3501822	0·3638581	0·4080920
5	0·1014902	0·0982047	0·0889864	0·3137959	0·3236347	0·3550024
6	—	0·0900283	0·0829645	—	0·2943270	0·3180441

From equations (6·1·9) the constants B_m, D_m may be found in terms of kB_0, $k'B_0$ and if these values are substituted in (6·1·7) two equations result for the three constants A_0, kB_0 and $k'B_0$. It must be remembered, however, that the identity (4·6·18) exists between some of the fundamental stress functions which are contained in (6·1·6) so that one of these three constants may be chosen at will. If k' is zero a preliminary calculation indicates that the values of D_m tend to constant multiples of kB_0 as m increases, while the values of B_m tend to zero. This form of solution is inconvenient as it makes the subsequent calculation of stresses very lengthy. The constant part of each D_m can be removed with the help of (4·6·18) so that the remaining part tends to zero as m increases. Unfortunately it is not easy to determine the constant part of D_m very accurately without including a large number of values of m in equations (6·1·9). An equivalent method is to choose the ratio of k and k' in (6·1·9) so that when the equations are solved for B_m and D_m all these constants continually decrease as m increases. There is evidently one ratio of $k:k'$ which will achieve this result but no analytical method for finding this ratio has been discovered. It has therefore been necessary to proceed by trial and error until a reasonably satisfactory falling off in the values of D_m were found although it is not possible to say explicitly that the values found tend to zero with m since they may tend to a numerically small constant.

The final values chosen for k and k' were $k = 1$, $k' = 0{\cdot}5$ and equations (6·1·9) were solved by successive approximations. The first approximation was found by ignoring all coefficients B_m, D_m for $m \geqslant 2$ and solving (6·1·9), when $r = 1$, for B_1, D_1. Using these values the equations, when $r = 2$, were then solved for B_2, D_2 ignoring B_m, D_m for $m \geqslant 3$. Then, when $r = 3$, B_3, D_3 were found by ignoring B_m, D_m for $m \geqslant 4$ and using the values already found for B_1, B_2,

D_1, D_2. This process was repeated until B_6, D_6 were obtained. A second approximation was then found by returning to the equations (6·1·9) for $r = 1$ and solving for B_1, D_1 using the first approximation values for $B_2, D_2, \ldots, B_6, D_6$ and ignoring the rest. Then when $r = 2$, B_2, D_2 were found by using the value just obtained for B_1, D_1 and the first approximation values for $B_3, D_3, \ldots, B_6, D_6$, ignoring the remaining constants. Similarly, second approximations were found for $B_3, D_3, \ldots, B_6, D_6$. Third and higher approximations can be found, by a similar process, the final values of the constants then being found by summing the separate approximations. This iterative process was found to converge quite rapidly so that it was unnecessary to proceed beyond the third approximation.

Values of B_m, D_m for $m = 1, 2, \ldots, 6$, together with the values of A_m which were obtained from (6·1·8), are given in table 6. These values of B_m, D_m were then substituted in (6·1·7) and the equations were solved for A_0 and B_0. The values found were

$$A_0 = 1{\cdot}69643\,T, \quad B_0 = 0{\cdot}158833\,T.$$

6·3. Chief interest lies in the value of the circumferential stress $\widehat{\theta\theta}$ at the edge of the hole $\rho = \lambda$. This has been found by evaluating $\widehat{rr}+\widehat{\theta\theta}$ at the hole by using the formulae which correspond to the stress function (6·1·6) and the values of the constants B_m, D_m given in table 6. Then, remembering that $\widehat{rr}$ is zero at the hole the value of $\widehat{\theta\theta}$ was found to be

$$\frac{\widehat{\theta\theta}}{T} = 1+\{-2{\cdot}0276-0{\cdot}1047\cos\pi\zeta+0{\cdot}0496\cos 2\pi\zeta-0{\cdot}0271\cos 3\pi\zeta$$
$$+0{\cdot}0168\cos 4\pi\zeta-0{\cdot}0113\cos 5\pi\zeta+0{\cdot}0081\cos 6\pi\zeta\}\cos 2\theta \quad (\lambda = 1). \quad (6{\cdot}3{\cdot}1)$$

It should be recalled that Poisson's ratio η has been taken to be 0·25.

TABLE 6

r	B_r	D_r	A_r
1	1·1943	0·77378	0·44974
2	−0·77132	−0·35831	−0·083340
3	0·46185	0·19451	0·027944
4	−0·29674	−0·11787	−0·012406
5	0·20395	0·07801	0·006519
6	−0·14817	−0·05583	−0·003834

It is difficult to give any precise information about the degree of accuracy of the coefficients in this Fourier expansion but it may be observed that when equations (6·1·9) were solved for the coefficients $B_1, D_1, \ldots, B_5, D_5$, all the remaining coefficients being zero, the value found for $\widehat{\theta\theta}$ agreed with the first six terms in the bracket in (6·3·1) or at most differed by one in the fourth decimal place. To be sure of three significant figures for the values of $\widehat{\theta\theta}$ it would be necessary to have more terms in the Fourier expansion but the considerable extra numerical work which would be required does not appear to be worth while, and it is only when all the terms in the bracket in (6·3·1) have the same sign that the third figure is likely to be much in error.

The value of $\widehat{zz}$ at the hole may most conveniently be found by evaluating $\widehat{rr}+\widehat{\theta\theta}+\widehat{zz}$ and then by using (6·3·1) and the fact that $\widehat{rr}$ is zero at the hole. This gives

$$\frac{\widehat{zz}}{T} = \{-0{\cdot}1689-0{\cdot}1211\cos\pi\zeta+0{\cdot}0293\cos 2\pi\zeta-0{\cdot}0105\cos 3\pi\zeta$$
$$+0{\cdot}0047\cos 4\pi\zeta-0{\cdot}0024\cos 5\pi\zeta+0{\cdot}0014\cos 6\pi\zeta\}\cos 2\theta \quad (\lambda = 1). \quad (6{\cdot}3{\cdot}2)$$

Some check on the accuracy of the work is provided by the fact that at the surfaces of the plate $\zeta = \pm 1$ the stress $\widehat{zz}$ should vanish. The formula (6·3·2) for $\widehat{zz}$ gives only a very small residual stress $0{\cdot}0005\,T\cos 2\theta$ when $\zeta = \pm 1$.

6·4. From (6·3·1) it is seen that the average value $\breve{\theta\theta}$ of $\widehat{\theta\theta}$ at a hole whose diameter is equal to the thickness of the plate changes from a *compression* $1{\cdot}028\,T$ at $\theta = 0°$ to a *tension* $3{\cdot}028\,T$ at $\theta = 90°$. The corresponding average values according to the theory of generalized plane stress are 1 and 3 respectively so that the approximate theory gives a good estimate of the average values of the circumferential stress at the hole in this case. The average value $\breve{zz}$ of $\widehat{zz}$ at the hole changes from a *compression* $0{\cdot}169\,T$ at $\theta = 0°$ to a *tension* $0{\cdot}169\,T$ at $\theta = 90°$ so that the neglection of $\breve{zz}$ compared with $\breve{\theta\theta}$ at the same points would hardly seem to be justified, especially at $\theta = 0°$.

The variation in $\widehat{\theta\theta}$ at $\theta = 0°$ and $90°$ across the thickness of the plate, calculated from (6·3·1), is shown in table 7. Across the middle sections of the plate at $\theta = 90°$ there is very little variation in the stress; there is rather more variation near the faces of the plate. The maximum stress is about $3{\cdot}10\,T$, i.e. just over 3 % in excess of the usually adopted value of $3\,T$, but to the order of accuracy given here it is not possible to say precisely whether the maximum occurs at the middle plane of the plate or at two points on either side of the middle plane. The value $2{\cdot}81\,T$ of $\widehat{\theta\theta}$ at the faces of the plate, which is probably a slight over-estimate of the actual value, is more than 6 % different from the generalized plane stress average value of $3\,T$. The percentage differences in the values of $\widehat{\theta\theta}$ at $\theta = 0°$ from the generalized plane stress average value of $-T$ are somewhat greater, varying from 10 % in excess numerically to 19 % below numerically.

TABLE 7

18ζ	$\widehat{\theta\theta}/T$ at $\theta = 0°$	$\widehat{\theta\theta}/T$ at $\theta = 90°$	18ζ	$\widehat{\theta\theta}/T$ at $\theta = 0°$	$\widehat{\theta\theta}/T$ at $\theta = 90°$
0	−1·10	3·10	10	−1·05	3·05
1	−1·10	3·10	11	−1·04	3·04
2	−1·10	3·10	12	−1·02	3·02
3	−1·10	3·10	13	−1·01	3·01
4	−1·09	3·09	14	−0·98	2·98
5	−1·09	3·09	15	−0·94	2·94
6	−1·08	3·08	16	−0·88	2·88
7	−1·08	3·08	17	−0·83	2·83
8	−1·08	3·08	18	−0·81	2·81
9	−1·07	3·07			

At $\theta = 0°$ and $\theta = 90°$ the maximum value of $\widehat{zz}$ is numerically about $0{\cdot}27\,T$ and occurs at the middle plane of the plate. It is a compression at $\theta = 0°$ and a tension at $\theta = 90°$. These values are by no means negligible when compared with $\widehat{\theta\theta}$ at the same points, particularly at $\theta = 0°$.

6·5. When a plate under tension in one direction contains a circular cylindrical hole whose diameter is equal to the thickness of the plate it appears that, although the assumptions of the theory of generalized plane stress are open to question the theory does give a fairly good estimate for the average values of stress concentration at the hole; also the maximum value of the circumferential stress at the hole, which is of considerable interest from the practical point of view, is only about 3 % in excess of the average value as given by the generalized plane stress theory. The circumferential stress $\widehat{\theta\theta}$ at the hole is almost constant in magnitude over the middle section of the plate, for a given value of θ, but there is an appreciable variation in $\widehat{\theta\theta}$ near the faces of the plate, the greatest variations occurring at the ends of the diameter of the hole which is parallel to the applied tension. In addition there is a significant cross stress $\widehat{zz}$ at certain points of the hole. The points at which the greatest differences between $\widehat{\theta\theta}$ and the generalized plane stress average value $\widetilde{\theta\theta}$ occur appear to be where $\widehat{zz}$ is not negligible compared with $\widehat{\theta\theta}$.

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References

Dougall, J. 1904 *Trans. Roy. Soc. Edinb.* **41**, 129.
Dougall, J. 1914 *Trans. Roy. Soc. Edinb.* **49**, 895.
Filon, L. N. G. 1903 *Phil. Trans.* A, **201**, 63.
Ghosh, S. 1946*a* *Bull. Calcutta Math. Soc.* **38**, 10.
Ghosh, S. 1946*b* *Bull. Calcutta Math. Soc.* **38**, 45.
Gray, A., Mathews, G. B. & MacRobert, T. M. 1922 *A treatise on Bessel functions*, 2nd ed. London: MacMillan.
Green, A. E. 1945 *Proc. Camb. Phil. Soc.* **41**, 224.
Newman, F. W. 1883 *Trans. Camb. Phil. Soc.* **13**, 145.
Southwell, R. V. 1936 *Phil. Mag.* (7), **21**, 201.
Watson, G. N. 1944 *A treatise on the theory of Bessel functions*, 2nd ed. Cambridge University Press.
Whittaker, E. & Robinson, G. 1946 *The calculus of observations*, 2nd ed. London: Blackie and Son.

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